8
Permutations

THE FACTORIAL

FACTORIAL NOTATION *Let n be a positive integer. Then, the continued product of first n natural numbers is called factorial n, to be denoted by n !, or* $\underline{|n}$ *. Also, we define* $0! = 1$.

When n is negative or a fraction, $n!$ is not defined.

Thus, $n! = n(n-1)(n-2)\ldots 3\cdot 2\cdot 1$.

EXAMPLE $6! = (6\times 5\times 4\times 3\times 2\times 1) = 720$, $3! = 3\times 2\times 1 = 6$ and $1! = 1$.

DEDUCTION $n! = n(n-1)(n-2)(n-3)\ldots 3\cdot 2\cdot 1$

$= n[(n-1)(n-2)(n-3)\ldots 3\cdot 2\cdot 1]$

$= n[(n-1)!]$.

Thus, $9! = 9\times(8!)$, $6! = 6\times(5!)$, $3! = 3\times(2!)$.

Also, $1! = 1\times(0!) \Rightarrow 0! = 1$.

Thus, we have

(i) $n! = n(n-1)(n-2)\ldots 3\times 2\times 1$.

(ii) $n! = n\times(n-1)!$.

(iii) $0! = 1$.

SOLVED EXAMPLES

EXAMPLE 1 *Compute:*

(i) $\dfrac{10!}{(7!)\times(3!)}$ *(ii)* $\dfrac{30!}{28!}$ *(iii)* $\dfrac{11!-10!}{9!}$

SOLUTION We have

(i) $\dfrac{10!}{(7!)\times(3!)} = \dfrac{10\times 9\times 8\times(7!)}{(7!)\times(3\times 2\times 1)} = \dfrac{10\times 9\times 8}{3\times 2\times 1} = 120.$

(ii) $\dfrac{30!}{28!} = \dfrac{30\times 29\times(28!)}{(28!)} = 30\times 29 = 870.$

(iii) $\dfrac{11!-10!}{9!} = \dfrac{11\times 10\times(9!)-10\times(9!)}{9!} = \dfrac{(110-10)(9!)}{(9!)} = 100.$

EXAMPLE 2 *If* $\dfrac{x}{10!} = \dfrac{1}{8!} + \dfrac{1}{9!}$, *find the value of x.*

SOLUTION On multiplying both sides by 10 !, we get

$$x = \frac{10!}{8!} + \frac{10!}{9!} \Rightarrow x = \frac{10 \times 9 \times (8!)}{(8!)} + \frac{10 \times (9!)}{(9!)}$$

$$\Rightarrow x = (10 \times 9) + 10 = (90 + 10) = 100.$$

Hence, $x = 100$.

EXAMPLE 3 *Express each of the following products in factorials:*

(i) $5 \times 6 \times 7 \times 8 \times 9$ *(ii)* $2 \times 4 \times 6 \times 8 \times 10$

SOLUTION We have

(i) $5 \times 6 \times 7 \times 8 \times 9 = \frac{(1 \times 2 \times 3 \times 4) \times (5 \times 6 \times 7 \times 8 \times 9)}{(1 \times 2 \times 3 \times 4)}$

[multiplying and dividing the given product by $(1 \times 2 \times 3 \times 4)$]

$$= \frac{1 \times 2 \times 3 \times 4 \times 5 \times 6 \times 7 \times 8 \times 9}{1 \times 2 \times 3 \times 4} = \frac{9!}{4!}.$$

(ii) $2 \times 4 \times 6 \times 8 \times 10 = (2 \times 1) \times (2 \times 2) \times (2 \times 3) \times (2 \times 4) \times (2 \times 5)$

$$= 2^5 \times (1 \times 2 \times 3 \times 4 \times 5) = 2^5 \times (5!).$$

EXAMPLE 4 *Find the LCM of* $(5!, 6!, 7!)$.

SOLUTION We have

$$\text{LCM}\,(5!, 6!, 7!) = \text{LCM}\,(5!, 6 \times 5!, 7 \times 6 \times 5!)$$

$$= (5!) \times \text{LCM}\,(1, 6, 42) = (5!) \times 42$$

$$= 7 \times 6 \times 5! = 7!.$$

EXAMPLE 5 *If* $(n+1)! = 90 \times (n-1)!$, *find n.*

SOLUTION We have

$$(n+1)! = 90 \times (n-1)!$$

$$\Rightarrow (n+1) \times n \times (n-1)! = 90 \times (n-1)!$$

$$\Rightarrow (n+1)n = 90$$

$$\Rightarrow (n+1)n = 10 \times 9$$

[writing 90 as product of two consecutive integers]

$$\Rightarrow n = 9.$$

Hence, $n = 9$.

EXAMPLE 6 *If* $(n+2)! = 60 \times (n-1)!$, *find n.*

SOLUTION We have

$$(n+2)! = 60 \times (n-1)!$$

$$\Rightarrow (n+2) \times (n+1) \times n \times (n-1)! = 60 \times (n-1)!$$

$$\Rightarrow (n+2)(n+1)n = 60$$

$$\Rightarrow (n+2)(n+1)n = 5 \times 4 \times 3$$

[writing 60 as product of 3 consecutive integers]

$$\Rightarrow n = 3.$$

Hence, $n = 3$.

EXAMPLE 7 *Prove that* $(n!+1)$ *is not divisible by any natural number between 2 and n.*

SOLUTION Since $n! = n(n-1)(n-2)\ldots 3\cdot 2\cdot 1$, it follows that $n!$ is divisible by every natural number between 2 and *n*. So, $(n!+1)$, when divided by any natural number between 2 and *n*, leaves 1 as remainder.

Hence, $(n!+1)$ is not divisible by any natural number between 2 and *n*.

EXAMPLE 8 *Prove that* $n(n-1)(n-2)\ldots(n-r+1) = \dfrac{n!}{(n-r)!}$.

SOLUTION We have

$$n(n-1)(n-2)\ldots(n-r+1)$$
$$= \frac{n(n-1)(n-2)\ldots(n-r+1)\cdot(n-r)!}{(n-r)!}$$
[multiplying num. and denom. by $(n-r)!$]
$$= \frac{n!}{(n-r)!}.$$

Hence, $n(n-1)(n-2)\ldots(n-r+1) = \dfrac{n!}{(n-r)!}$.

EXAMPLE 9 *Prove that* $\dfrac{(2n)!}{n!} = 2^n \times \{1\times 3\times 5\times \ldots \times (2n-1)\}$.

SOLUTION We have

$$\frac{(2n)!}{n!} = \frac{\{1\times 2\times 3\times 4\times 5\times \ldots\times (2n-2)\times (2n-1)\times 2n\}}{n!}$$
$$= \frac{\{1\times 3\times 5\times \ldots\times (2n-1)\}\times\{2\times 4\times 6\times \ldots\times (2n-2)\times 2n\}}{n!}$$
$$= \frac{\{1\times 3\times 5\times \ldots\times (2n-1)\}\times 2^n\times\{1\times 2\times 3\times \ldots\times n\}}{n!}$$
$$= \frac{\{1\times 3\times 5\times \ldots\times (2n-1)\}\times 2^n\times (n!)}{n!}$$
$$= 2^n\times\{1\times 3\times 5\times \ldots\times (2n-1)\}.$$

Hence, $\dfrac{(2n)!}{n!} = 2^n\times\{1\times 3\times 5\times \ldots\times (2n-1)\}$.

EXAMPLE 10 *Prove that* $(n!)^2 \le (n!)\times n^n < (2n)!$ *for all* $n \in N$.

SOLUTION We have

$$(n!)^2 = (n!)\times(n!) = (n!)\times(1\times 2\times 3\times \ldots\times n)$$
$$\le (n!)\times n^n \qquad [\because\ r \le n \text{ for each } r = 1, 2, 3, \ldots, n].$$
$$\therefore\quad (n!)^2 \le (n!)\times n^n. \qquad \ldots \text{(i)}$$

Again, we have

$$(2n)! = (1\times 2\times 3\times \ldots\times n)\times(n+1)\times(n+2)\times \ldots\times 2n$$
$$> (n!)\times n^n. \qquad [\because\ (n+r) > n \text{ for each } r = 1, 2, \ldots, n].$$

Thus, $(n!)\times n^n < (2n)!$. … (ii)

Hence, from (i) and (ii), we have

$$(n!)^2 \le (n!)\times n^n < (2n)!.$$

EXERCISE 8A

1. Compute:

(i) $\frac{9!}{(5!)\times(3!)}$ (ii) $\frac{32!}{29!}$ (iii) $\frac{(12!)-(10!)}{9!}$

2. Prove that LCM $\{6!, 7!, 8!\} = 8!$.

3. Prove that $\frac{1}{10!}+\frac{1}{11!}+\frac{1}{12!}=\frac{145}{12!}$.

4. If $\frac{1}{6!}+\frac{1}{7!}=\frac{x}{8!}$, find the value of x.

5. Write the following products in factorial notation:

(i) $6\times7\times8\times9\times10\times11\times12$ (ii) $3\times6\times9\times12\times15$

6. Which of the following are true or false?

(i) $(2+3)!=2!+3!$ (ii) $(2\times3)!=(2!)\times(3!)$

7. If $(n+1)!=12\times(n-1)!$, find the value of n.

8. If $(n+2)!=2550\times n!$, find the value of n.

9. If $(n+3)!=56\times(n+1)!$, find the value of n.

10. If $\frac{n!}{(2!)\times(n-2)!}:\frac{n!}{(4!)\times(n-4)!}=2:1$, find the value of n.

11. If $\frac{(2n)!}{(3!)\times(2n-3)!}:\frac{n!}{(2!)\times(n-2)!}=44:3$, find the value of n.

12. Evaluate $\frac{n!}{(r!)\times(n-r)!}$, when $n=15$ and $r=12$.

13. Prove that $(n+2)\times(n!)=(n!)+(n+1)!$.

14. Prove that:

(i) $\frac{n!}{r!}=n(n-1)(n-2)\ldots(r+1)$

(ii) $(n-r+1)\cdot\frac{n!}{(n-r+1)!}=\frac{n!}{(n-r)!}$

(iii) $\frac{n!}{r!(n-r)!}+\frac{n!}{(r-1)!(n-r+1)!}=\frac{(n+1)!}{r!(n-r+1)!}$

ANSWERS (EXERCISE 8A)

1. (i) 504 (ii) 29760 (iii) 1310 **4.** $x=64$

5. (i) $\frac{12!}{5!}$ (ii) $3^5(5!)$ **6.** (i) False (ii) False **7.** $n=3$

8. $n=49$ **9.** $n=5$ **10.** $n=5$ **11.** $n=6$ **12.** 455

HINTS TO SOME SELECTED QUESTIONS

3. LHS $=\left\{\frac{1}{10!}+\frac{1}{11\times(10!)}\right\}+\frac{1}{12!}=\frac{(11+1)}{11\times(10!)}+\frac{1}{12\times(11!)}$

$$= \frac{12}{11!} + \frac{1}{12 \times (11!)} = \frac{(144+1)}{12 \times (11!)} = \frac{145}{12!} = \text{RHS}.$$

7. $(n+1)n \times (n-1)! = 12 \times (n-1)! \Rightarrow (n+1) \times n = 12 = 4 \times 3 \Rightarrow n = 3.$

8. $(n+2)(n+1) \times (n!) = 51 \times 50 \times (n!) \Rightarrow (n+2)(n+1) = 51 \times 50$

$\Rightarrow n+1 = 50 \Rightarrow n = 49.$

9. $(n+3)(n+2) \times (n+1)! = 8 \times 7 \times (n+1)! \Rightarrow (n+3)(n+2) = 8 \times 7 \Rightarrow n+2 = 7 \Rightarrow n = 5.$

10. $\frac{n!}{2 \times (n-2)(n-3) \times (n-4)!} \times \frac{24 \times (n-4)!}{n!} = \frac{2}{1}$

$\Rightarrow \quad (n-2)(n-3) = 6 = 3 \times 2 \Rightarrow n-3 = 2 \Rightarrow n = 5.$

11. $\frac{(2n)(2n-1)(2n-2) \times (2n-3)!}{6 \times (2n-3)!} \times \frac{2 \times (n-2)!}{n(n-1) \times (n-2)!} = \frac{44}{3}$

$\Rightarrow \quad \frac{8 \times n \times (n-1) \times (2n-1)}{6 \times n \times (n-1)} = \frac{44}{3} \Rightarrow 2n-1 = \frac{44}{3} \times \frac{6}{8} \Rightarrow 2n-1 = 11 \Rightarrow n = 6.$

12. $\frac{15!}{(12!) \times (3!)} = \frac{15 \times 14 \times 13 \times (12!)}{6 \times (12!)} = 455.$

13. $\text{RHS} = (n!) + (n+1) \times (n!) = (n+2) \times (n!) = \text{LHS}.$

14. (i) $\text{RHS} = \frac{n(n-1)(n-2)\ldots(r+1) \times (r!)}{r!} = \frac{n!}{r!}.$

(ii) We know that $(n-r+1)! = (n-r+1) \cdot (n-r)!.$

(iii) $\text{LHS} = \frac{n!}{(n-r)! \cdot (r-1)!} \left\{ \frac{1}{r} + \frac{1}{n-r+1} \right\}$

$= \frac{n!}{(n-r)! \cdot (r-1)!} \cdot \frac{(n+1)}{r(n-r+1)} = \frac{(n+1)!}{(r!) \cdot (n-r+1)!} = \text{RHS}.$

FUNDAMENTAL PRINCIPLES OF COUNTING

1. FUNDAMENTAL PRINCIPLE OF MULTIPLICATION *If there are two operations such that one of them can be performed in m ways, and when it has been performed in anyone of these m ways, the second operation can be performed in n ways then the two operations in succession can be performed in $(m \times n)$ ways.*

EXAMPLE 1 *In a class there are 15 boys and 12 girls. The teacher wants to select 1 boy and 1 girl to represent the class for a function. In how many ways can the teacher make the selection?*

SOLUTION Here the teacher is to perform two operations:

(i) selecting 1 boy out of 15 boys,

and (ii) selecting 1 girl out of 12 girls.

The first of these can be done in 15 ways and the second in 12 ways.

So, by the fundamental principle of multiplication, the required number of ways $= (15 \times 12) = 180.$

Hence, the teacher can make the selection of 1 boy and 1 girl in 180 ways.

2. FUNDAMENTAL PRINCIPLE OF ADDITION *If there are two operations such that they can be performed independently in m and n ways respectively then either of the two operations can be performed in $(m+n)$ ways.*

EXAMPLE 2 *In a class there are 16 boys and 9 girls. The teacher wants to select either a boy or a girl as a class representative. In how many ways can the teacher make the selection?*

SOLUTION Here the teacher is to perform either of the following two operations:

(i) selecting 1 boy out of 16 boys,

and (ii) selecting 1 girl out of 9 girls.

The first of these can be performed in 16 ways and the second in 9 ways.

By the fundamental principle of addition, either of the two operations can be performed in $(16+9)$ ways = 25 ways.

Hence, the teacher can make the selection of 1 boy or 1 girl in 25 ways.

ILLUSTRATIVE EXAMPLES

EXAMPLE 1 *In a cinema hall, there are three entrance doors and two exit doors. In how many ways can a person enter the hall and then come out?*

SOLUTION Clearly, a person can enter the hall through any of the three entrance doors. So, there are 3 ways of entering the hall.

After entering the hall, the person can come out through any of the two exit doors. So, there are 2 ways of coming out.

Hence, by the fundamental principle of multiplication, the number of ways in which a person can enter the hall and then come out $= (3 \times 2) = 6$.

EXAMPLE 2 *Three persons enter a railway carriage, where there are 5 vacant seats. In how many ways can they seat themselves?*

SOLUTION Clearly, the first person can occupy any of the 5 seats.

So, there are 5 ways in which the first person can seat himself.

Now, the second person can occupy any of the remaining 4 seats. So, the second person can be seated in 4 ways.

Similarly, the third person can occupy a seat in 3 ways.

Hence, by the fundamental principle of multiplication, the required number of ways $= (5 \times 4 \times 3) = 60$.

EXAMPLE 3 *The flag of a newly formed forum is in the form = = = of three blocks, each to be coloured differently. If there are six different colours on the whole to choose from, how many such designs are possible?*

SOLUTION The first block of the flag can be coloured by anyone of the 6 given colours. So, there are 6 ways to colour the first block.

Now, the second block can be coloured by anyone of the remaining five colours. So, there are 5 ways to colour the second block.

The third block can now be coloured by anyone of the remaining four colours. So, there are 4 ways to colour the third block.

Hence, by the fundamental principle of multiplication, the required number of designs $= (6 \times 5 \times 4) = 120$.

EXAMPLE 4 *There are 4 routes between Delhi and Patna. In how many different ways can a man go from Delhi to Patna and return, if for returning*

(i) any of the routes is taken;

(ii) the same route is taken;

(iii) the same route is not taken?

SOLUTION We have the following three cases.

Case (i) *When any of the routes is taken for returning:*

The man may take any route for going from Delhi to Patna.

So, there are 4 ways of going from Delhi to Patna.

When done so, he may return by any of the 4 routes.

So, there are 4 ways of returning from Patna to Delhi.

Hence, by the fundamental principle of multiplication, the total number of ways for going to Patna and returning back to Delhi $= (4 \times 4) = 16$.

Case (ii) *When the same route is taken for returning*:

In this case, there are 4 ways of going to Patna and only 1 way of returning, namely by the same route.

Hence, the required number of ways $= (4 \times 1) = 4$.

Case (iii) *When the same route is not taken for returning:*

In this case, there are 4 ways of going to Patna.

But, the man does not return by the same route.

So, there are 3 ways of returning back to Delhi.

Hence, the required number of ways $= (4 \times 3) = 12$.

EXAMPLE 5 *There are six multiple-choice questions in an examination. Find the total number of ways of answering these questions, if the first three questions have 5 choices each and the next three have 4 choices each.*

SOLUTION Each of the first three questions has 5 choices.

So, each one of them can be answered in 5 ways.

Each of the next three questions has 4 choices.

So, each one of them can be answered in 4 ways.

Hence, by the fundamental principle of multiplication, the total number of ways of answering the six questions $= (5 \times 5 \times 5 \times 4 \times 4 \times 4) = 8000$.

EXAMPLE 6 *Four flags of different colours are given. How many different signals can be generated, if a signal requires the use of two flags, one below the other?*

SOLUTION The total number of signals is equal to the number of ways of filling two places $\boxed{}$ in succession by four flags of different colours.

The upper place can be filled in 4 different ways by any of the four flags. Following it, the lower space can be filled in 3 different ways by anyone of the remaining three flags.

Hence, by the fundamental principle of multiplication, the required number of signals $= (4 \times 3) = 12$.

EXAMPLE 7 *Find the number of 4-letter words, with or without meaning, which can be formed out of the letters of the word, 'NOSE', when:*

(i) the repetition of the letters is not allowed,

(ii) the repetition of the letters is allowed.

SOLUTION **Case (i)** *When the repetition of the letters is not allowed:*

In this case, the total number of words is the same as the number of ways of filling 4 places $____$ by 4 different letters chosen from N, O, S, E.

The first place can be filled by any of the 4 letters. Thus, there are 4 ways of filling the first place. Since the repetition of letters is not allowed, so the second place can be filled by any of the remaining 3 letters in 3 different ways.

Following it, the third place can be filled by any of the two remaining letters in 2 different ways. And, the fourth place can be filled by the remaining one letter in 1 way.

So, by the fundamental principle of multiplication, the required number of 4-letter words $= (4 \times 3 \times 2 \times 1) = 24$.

Hence, required number of words = 24.

Case (ii) *When the repetition of the letters is allowed*:

In this case, each of the four different places can be filled in succession in 4 different ways.

So, the required number of 4-letter words $= (4 \times 4 \times 4 \times 4) = 256$.

EXAMPLE 8 *How many words (with or without meaning) of three distinct letters of the English alphabet are there?*

SOLUTION Clearly, we have to fill up three places by distinct letters of the English alphabet.

The first place can be filled by any of the 26 letters. Thus, there are 26 ways to fill the first place.

The second place can be filled by any of the remaining 25 letters. So, there are 25 ways to fill the second place.

The third place can be filled by any of the remaining 24 letters.

Thus, there are 24 ways to fill the third place.

Hence, by the fundamental principle of multiplication, the required number of words = $(26 \times 25 \times 24) = 15600$.

EXAMPLE 9 *How many numbers are there between* 100 *and* 1000 *in which all the digits are distinct?*

SOLUTION Clearly, every number between 100 and 1000 is a 3-digit number.

So, we have to form all possible 3-digit numbers with distinct digits. We cannot have 0 at the hundred's place.

So, the hundred's place can be filled with any of the nine digits, namely 1, 2, 3, 4, 5, 6, 7, 8, 9.

So, there are 9 ways of filling the hundred's place. After filling the hundred's place, nine digits are left including 0. So, there are 9 ways of filling the ten's place.

Now, the unit's place can be filled by any of the remaining 8 digits. So, there are 8 ways of filling the unit's place.

Hence, the required number of numbers = $(9 \times 9 \times 8) = 648$.

EXAMPLE 10 *How many* 9*-digit numbers of different digits can be formed?*

SOLUTION The ten-crore's place of a 9-digit number cannot be 0.

So, this place can be filled up by any of the digits from 1 to 9. So, there are 9 ways of filling the ten-crore's place.

The crore's place can now be filled with 0 or any of the remaining 8 digits. Thus, there are 9 ways of filling it.

The ten-lakh's place can now be filled with any of the remaining 8 digits. Thus, there are 8 ways of filling it.

Similarly, we can fill lakh's place in 7 ways; ten thousand's place in 6 ways; thousand's place in 5 ways; hundred's place in 4 ways; ten's place in 3 ways and unit's place in 2 ways.

Hence, by the fundamental principle of multiplication, the required number of numbers = $(9 \times 9 \times 8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2) = 3265920$.

EXAMPLE 11 *How many numbers between* 2000 *and* 3000 *can be formed from the digits* 2, 3, 4, 5, 6, 7 *when repetition of digits is not allowed?*

SOLUTION Clearly, a number between 2000 and 3000 will have 2 at the thousand's place. So, this place can be filled in 1 way only.

Now, the hundred's place can be filled by any of the remaining five digits. So, there are 5 ways of filling the hundred's place.

Similarly, the ten's place can be filled in 4 ways and the unit's place can be filled in 3 ways.

Hence, by the fundamental principle of multiplication, the total number of required numbers = $(1 \times 5 \times 4 \times 3) = 60$.

EXAMPLE 12 *How many* 3*-digit odd numbers can be formed by using the digits* 1, 2, 3, 4, 5, 6 *when*

(i) the repetition of digits is not allowed?

(ii) the repetition of digits is allowed?

SOLUTION For a number to be odd, we must have 1, 3 or 5 at the unit's place. So, there are 3 ways of filling the unit's place.

Case (i) *When the repetition of digits is not allowed*:

In this case, after filling the unit's place, we may fill the ten's place by any of the remaining five digits. So, there are 5 ways of filling the ten's place.

Now, the hundred's place can be filled by any of the remaining 4 digits. So, there are 4 ways of filling the hundred's place.

So, by the fundamental principle of multiplication, the required number of odd numbers = $(3 \times 5 \times 4) = 60$.

Case (ii) *When the repetition of digits is allowed:*

Since the repetition of digits is allowed, so after filling the unit's place, we may fill the ten's place by any of the given six digits. So, there are 6 ways of filling the ten's place.

Similarly, the hundred's place can be filled by any of the given six digits. So, it can be filled in 6 ways.

Hence, by the fundamental principle of multiplication, the required number of odd numbers = $(3 \times 6 \times 6) = 108$.

EXAMPLE 13 *How many odd numbers less than* 1000 *can be formed by using the digits* 0, 2, 5, 7 *when the repetition of digits is allowed?*

SOLUTION Since each number is less than 1000, required numbers are the 1-digit, 2-digit and 3-digit numbers.

One-digit numbers: Clearly, there are two one-digit odd numbers, namely 5 and 7, formed of the given digits.

Two-digit numbers: Since we are to form 2-digit odd numbers, we may put 5 or 7 at the unit's place. So, there are 2 ways of filling the unit's place.

Now, we cannot use 0 at the ten's place and the repetition of digits is allowed. So, we may fill up the ten's place by any of the digits 2, 5, 7. Thus, there are 3 ways of filling the ten's place.

Hence, the required type of 2-digit numbers = $(2 \times 3) = 6$.

Three-digit numbers: To have an odd 3-digit number, we may put 5 or 7 at the unit's place. So, there are 2 ways of filling the unit's place.

We may fill up the ten's place by any of the digits 0, 2, 5, 7. So, there are 4 ways of filling the ten's place.

We cannot put 0 at the hundred's place. So, the hundred's place can be filled by any of the digits 2, 5, 7 and so it can be done in 3 ways.

$\therefore$ the required number of 3-digit numbers = $(2 \times 4 \times 3) = 24$.

Hence, the total number of required type of numbers $= (2 + 6 + 24) = 32$.

EXAMPLE 14 *How many numbers are there between* 100 *and* 1000 *such that* 7 *is in the unit's place?*

SOLUTION Clearly, the numbers between 100 and 1000 are 3-digit numbers.

So, we have to form 3-digit numbers with 7 at the unit's place.

Clearly, 0 cannot be there at the hundred's place.

So, the hundred's place can be filled with any of the digits from 1 to 9.

Thus, the hundred's place can be filled in 9 ways.

Now, the ten's place can be filled with any of the digits from 0 to 9.

So, there are 10 ways of filling the ten's place.

The unit's place can be filled with 7, i.e., in 1 way only.

Hence, the number of required type of numbers $= (9 \times 10 \times 1) = 90$.

EXAMPLE 15 *How many numbers are there between* 100 *and* 1000 *such that at least one of their digits is* 7*?*

SOLUTION Clearly, the numbers between 100 and 1000 are 3-digit numbers.

So, we have to form 3-digit numbers such that at least one of their digits is 7.

Case I *Three-digit numbers with* 7 *at unit's place:*

The number of ways to fill the hundred's place = 9 [by any digit from 1 to 9].

The number of ways to fill the ten's place = 10 [by any digit from 0 to 9].

The number of ways to fill the unit's place = 1 [by 7 only].

$\therefore$ the number of such numbers $= (9 \times 10 \times 1) = 90$.

Case II *Three-digit numbers with* 7 *at ten's place:*

The number of ways to fill the hundred's place = 9 [by any digit from 1 to 9].

The number of ways to fill the ten's place = 1 [by 7 only].

The number of ways to fill the unit place = 10 [by any digit from 0 to 9].

$\therefore$ the number of such numbers $= (9 \times 1 \times 10) = 90$.

Case III *Three-digit numbers with* 7 *at hundred's place:*

The number of ways to fill the hundred's place = 1 [by 7 only].

The number of ways to fill the ten's place = 10 [by any digit from 0 to 9].

The number of ways to fill the unit's place = 10 [by any digit from 0 to 9].

$\therefore$ the number of such numbers $= (1 \times 10 \times 10) = 100.$

Hence, the total number of required numbers $= (90 + 90 + 100) = 280.$

EXAMPLE 16 *How many numbers are there between* 100 *and* 1000*, which have exactly one of their digits as* 7*?*

SOLUTION Clearly, the numbers between 100 and 1000 are 3-digit numbers.

So, we have to form 3-digit numbers having exactly one of their digits as 7.

Case I 3-*digit numbers having* 7 *at unit's place only:*

Number of ways to fill the unit's place $= 1$ [with 7 only].

Number of ways to fill the ten's place $= 9$ [any digit from 0 to 9 except 7].

Number of ways to fill the hundred's place $= 8$ [any digit from 1 to 9 except 7].

So, the number of such numbers $= (1 \times 9 \times 8) = 72.$

Case II 3-*digit numbers having* 7 *at ten's place only:*

Number of ways to fill the ten's place $= 1$ [with 7 only].

Number of ways to fill the unit's place $= 9$ [any digit from 0 to 9 except 7].

Number of ways to fill the hundred's place $= 8$ [any digit from 1 to 9 except 7].

So, the number of such numbers $= (1 \times 9 \times 8) = 72.$

Case III 3-*digit numbers having* 7 *at hundred's place only:*

Number of ways to fill the hundred's place $= 1$ [with 7 only].

Number of ways to fill the ten's place $= 9$ [any digit from 0 to 9 except 7].

Number of ways to fill the unit's place $= 9$ [any digit from 0 to 9 except 7].

So, the number of such numbers $= (1 \times 9 \times 9) = 81.$

Hence, the total number of required numbers $= (72 + 72 + 81) = 225.$

EXAMPLE 17 *How many numbers are there between* 100 *and* 1000 *such that every digit is either* 2 *or* 9*?*

SOLUTION Clearly, the numbers between 100 and 1000 are 3-digit numbers.

So, we have to form 3-digit numbers by using 2 and 9.

Clearly, the repetition of digits is allowed.

Each one of the unit's, ten's and hundred's places can be filled in 2 ways.

Hence, the required number of numbers $= (2 \times 2 \times 2) = 8.$

EXAMPLE 18 *How many* 3*-digit numbers can be formed without using the digits* 0, 2, 3, 4, 5 *and* 6*?*

SOLUTION Clearly, we have to form 3-digit numbers by using the digits 1, 7, 8 and 9 while the repetition of digits is allowed.

Clearly, each one of the unit's, ten's and hundred's places can be filled by any of the digits 1, 7, 8, 9, i.e., in four ways.

Hence, the number of required numbers $= (4 \times 4 \times 4) = 64$.

EXERCISE 8B

1. There are 10 buses running between Delhi and Agra. In how many ways can a man go from Delhi to Agra and return by a different bus?
2. A, B and C are three cities. There are 5 routes from A to B and 3 routes from B to C. How many different routes are there from A to C via B?
3. There are 12 steamers plying between A and B. In how many ways could the round trip from A be made if the return was made on (i) the same steamer? (ii) a different steamer?
4. In how many ways can 4 people be seated in a row containing 6 seats?
5. In how many ways can 5 ladies draw water from 5 taps, assuming that no tap remains unused?
6. In a textbook on mathematics there are three exercises A, B and C consisting of 12, 18 and 10 questions respectively. In how many ways can three questions be selected choosing one from each exercise?
7. In a school, there are four sections of 40 students each in XI standard. In how many ways can a set of 4 student representatives be chosen, one from each section?
8. In how many ways can a vowel, a consonant and a digit be chosen out of the 26 letters of the English alphabet and the 10 digits?
9. How many 8-digit telephone numbers can be constructed using the digits 0 to 9 if each number starts with 270 and no digit appears more than once?
10. (a) A coin is tossed three times and the outcomes are recorded. How many possible outcomes are there?

    (b) How many possible outcomes if the coin is tossed:

    (i) four times? (ii) five times? (iii) n times?
11. Find the number of different signals that can be generated by arranging at least 2 flags in order (one below the other) on a vertical staff, if five different flags are available.
12. How many 4-letter codes can be formed using the first 10 letters of the English alphabet, if no letter can be repeated?
13. Given, $A = \{2, 3, 5\}$ and $B = \{0, 1\}$. Find the number of different ordered pairs in which the first entry is an element of A and the second is an element of B.
14. How many arithmetic progressions with 10 terms are there whose first term is in the set {1, 2, 3} and whose common difference is in the set {2, 3, 4}?

15. There are 6 items in column A and 6 items in column B. A student is asked to match each item in column A with an item in column B. How many possible (correct or incorrect) answers are there to this question?
16. A mint prepares metallic calendars specifying months, dates and days in the form of monthly sheets (one plate for each month). How many types of February calendars should it prepare to serve for all the possibilities in the future years?
17. From among the 36 teachers in a school, one principal and one vice-principal are to be appointed. In how many ways can this be done?
18. A sample of 3 bulbs is tested. A bulb is labelled as *G* if it is good and *D* if it is defective. Find the number of all possible outcomes.
19. For a set of five true or false questions, no student has written the all correct answer and no two students have given the same sequence of answers. What is the maximum number of students in the class for this to be possible?
20. In how many ways can the following prizes be given away to a class of 20 students:

    first and second in mathematics; first and second in chemistry; first in physics and first in English?
21. Find the total number of ways of answering 5 objective-type questions, each question having 4 choices.
22. A gentleman has 6 friends to invite. In how many ways can he send invitation cards to them, if he has 3 servants to carry the cards?
23. In how many ways 6 rings of different types can be worn in 4 fingers?
24. In how many ways can 5 letters be posted in 4 letter boxes?
25. How many 3-letter words can be formed using a, b, c, d, e if
    (i) repetition of letters is not allowed?
    (ii) repetition of letters is allowed?
26. How many 4-digit numbers are there, when a digit may be repeated any number of times?
27. How many numbers can be formed from the digits 1, 3, 5, 9 if repetition of digits is not allowed?
28. How many 3-digit numbers are there with no digit repeated?
29. How many 3-digit numbers can be formed by using the digits 0, 1, 3, 5, 7 while each digit may be repeated any number of times?
30. How many 6-digit numbers can be formed from the digits 0, 1, 3, 5, 7, 9 when no digit is repeated? How many of them are divisible by 10?
31. How many natural numbers less than 1000 can be formed from the digits 0, 1, 2, 3, 4, 5 when a digit may be repeated any number of times?
32. How many 6-digit telephone numbers can be constructed using the digits 0 to 9, if each number starts with 67 and no digit appears more than once?

33. In how many ways can three jobs I, II and III be assigned to three persons A, B and C if one person is assigned only one job and all are capable of doing each job?

34. A number lock on a suitcase has three wheels each labelled with ten digits 0 to 9. If opening of the lock is a particular sequence of three digits with no repeats, how many such sequences will be possible? Also, find the number of unsuccessful attempts to open the lock.

35. A customer forgets a four-digit code for an automated teller machine (ATM) in a bank. However, he remembers that this code consists of digits 3, 5, 6, 9. Find the largest possible number of trials necessary to obtain the correct code.

36. In how many ways can 3 prizes be distributed among 4 girls, when

(i) no girl gets more than one prize?

(ii) a girl may get any number of prizes?

(iii) no girl gets all the prizes?

ANSWERS (EXERCISE 8B)

1. 90	**2.** 15	**3.** (i) 12 (ii) 132	**4.** 360	**5.** 120	**6.** 2160
7. 2560000	**8.** 1050	**9.** 2520	**10.** (a) 8 (b) (i) 16 (ii) 32 (iii) 2^n		
11. 320	**12.** 5040	**13.** 6	**14.** 9	**15.** 720	**16.** 14
17. 1260	**18.** 8	**19.** 31	**20.** 57760000		**21.** 4^5
22. 3^6	**23.** 4^6	**24.** 4^5	**25.** (i) 60 (ii) 125		**26.** 9000
27. 64	**28.** 648	**29.** 100	**30.** 600, 120		**31.** 215
32. 1680	**33.** 6	**34.** 720, 719	**35.** 24	**36.** (i) 24 (ii) 64 (iii) 60	

HINTS TO SOME SELECTED QUESTIONS

3. (i) In this case the round trip from A is to be made by the same steamer. So, there are 12 ways of making the round trips.

(ii) In this case, number of ways to go from A to B = 12.

Number of ways to go from B to A = 11.

Hence, the required number of ways = $(12 \times 11) = 132$.

4. 1st person can be seated in 6 ways, 2nd person in 5 ways, 3rd person in 4 ways and 4th person in 3 ways.

Required number of ways = $(6 \times 5 \times 4 \times 3) = 360$.

5. 1st lady can draw water from any of the 5 taps.

So, first lady can draw water in 5 ways, 2nd in 4 ways, 3rd in 3 ways, 4th in 2 ways and 5th in 1 way.

Required number of ways = $(5 \times 4 \times 3 \times 2 \times 1) = 120$.

6. 1 question out of 12 can be chosen in 12 ways.

1 question out of 18 can be chosen in 18 ways.

1 question out of 10 can be chose in 10 ways.

Required number of ways = $(12 \times 18 \times 10) = 2160$.

7. From each of the 4 sections 1 representative out of 40 can be chosen in 40 ways.

So, by the fundamental principle of multiplication, the required number of ways $= (40 \times 40 \times 40 \times 40) = 2560000.$

8. There are 5 vowels, 21 consonants and 10 digits.

Number of ways of choosing 1 vowel out of 5 vowels = 5.

Number of ways of choosing 1 consonant out of 21 consonants = 21.

Number of ways of choosing 1 digit out of 10 digits = 10.

Required number of ways $= (5 \times 21 \times 10) = 1050.$

9. The remaining digits to be used are 1, 3, 4, 5, 6, 8, 9.

And, we have to select 5 digits out of these eight.

First digit can be anyone of these 7 digits.

So, there are 7 ways of choosing the 1st digit.

Similarly, we may choose 2nd digit in 6 ways, 3rd digit in 5 ways, 4th digit in 4 ways and 5th digit in 3 ways.

By the fundamental principle of multiplication, the required number of ways $= (7 \times 6 \times 5 \times 4 \times 3) = 2520.$

10. (a) In a single throw of a coin there are 2 possible outcomes namely Head (H) and Tail (T).

So, by the fundamental principle of multiplication, the total number of possible outcomes in 3 tosses $= (2 \times 2 \times 2) = 2^3 = 8.$

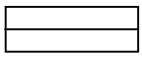
(b) Similarly, we have

(i) total number of possible outcomes in 4 tosses $= (2 \times 2 \times 2 \times 2) = 2^4 = 16.$

(ii) total number of possible outcomes in 5 tosses $= (2 \times 2 \times 2 \times 2 \times 2) = 2^5 = 32.$

(iii) total number of possible outcomes in n tosses $= 2^n.$

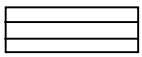
11. A signal can consist of either 2 flags, 3 flags, 4 flags or 5 flags.

2-flag signals: 

1st place can be filled by any of the 5 flags in 5 ways.

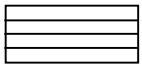
2nd place can be filled by any of the remaining 4 flags in 4 ways.

Total number of 2-flag signals $= (5 \times 4) = 20.$

3-flag signals: 

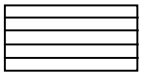
1st place can be filled in 5 ways, 2nd in 4 ways and 3rd in 3 ways.

Total number of 3-flag signals $= (5 \times 4 \times 3) = 60.$

4-flag signals: 

1st place can be filled in 5 ways, 2nd in 4 ways, 3rd in 3 ways and 4th in 2 ways.

Total number of 4-flag signals $= (5 \times 4 \times 3 \times 2) = 120.$

5-flag signals: 

1st place can be filled in 5 ways, 2nd in 4 ways, 3rd in 3 ways, 4th in 2 ways and 5th in 1 way only.

Total number of 5-flag signals $= (5 \times 4 \times 3 \times 2 \times 1) = 120.$

Hence, total number of signals $= (20 + 60 + 120 + 120) = 320.$

12. We have to fill up 4 places ==== by first 10 letters of the English alphabet. The first place can be filled by any of the 10 letters in 10 ways.

Similarly, 2nd place can be filled in 9 ways; 3rd place can be filled in 8 ways and 4th place can be filled in 7 ways.

Required number of 4-letter codes $= (10 \times 9 \times 8 \times 7) = 5040$.

13. If $n(A) = p$ and $n(B) = q$, then $n(A \times B) = pq$.

Here $n(A) = 3$ and $n(B) = 2$, then $n(A \times B) = (3 \times 2) = 6$.

14. Required number of AP's $= (3 \times 3) = 9$.

15. First item in column A can be matched with any of the 6 items in column B.

Thus, the first item of column A can be matched in 6 ways.

Similarly, 2nd item of column A can be matched in 5 ways.

Similarly, 3rd, 4th, 5th and 6th items of column A can be matched in 4, 3, 2 and 1 ways respectively.

By the fundamental principle of multiplication, the required number of ways $= (6 \times 5 \times 4 \times 3 \times 2 \times 1) = 720$.

16. There are two types of Februaries, one having 28 days and the other having 29 days. So, there are 2 ways of preparing calendars for February.

Also, the month may start from any day from Monday to Sunday. So, there are 7 ways of preparing it for various days.

Hence, the required number of plates $= (2 \times 7) = 14$.

17. Number of ways of choosing 1 principal out of 36 teachers = 36.

Number of ways of choosing 1 vice-principal out of 35 teachers = 35.

Hence, the required number of ways $= (36 \times 35) = 1260$.

18. All possible outcomes are

GGG, GGD, GDG, DGG, GDD, DGD, DDG, DDD.

Clearly, there are 8 possible outcomes.

19. Clearly there are 2 ways of answering each of the 5 questions, i.e., true or false.

So, the total number of different sequences of answers $= (2 \times 2 \times 2 \times 2 \times 2) = 32$.

There is only one all correct answer.

Maximum number of sequences of answers leaving aside the all correct answer $= (32 - 1) = 31$.

Hence, the maximum possible number of students = 31.

20. 1st and 2nd prizes in mathematics can be given in (20×19) ways;

1st and 2nd prizes in chemistry can be given in (20×19) ways;

1st prize in physics can be given in 20 ways;

1st prize in English can be given in 20 ways.

Required number of ways in all $= (20 \times 19) \times (20 \times 19) \times 20 \times 20 = 57760000$.

21. Since each question can be answered in 4 ways, total number of ways of answering 5 questions $= (4 \times 4 \times 4 \times 4 \times 4) = 4^5$.

22. Each friend can be invited by any of the 3 servants, i.e., in 3 ways. So, the required number of ways $= (3 \times 3 \times 3 \times 3 \times 3 \times 3) = 3^6$.

23. The first ring can be worn in any of the 4 fingers. So, there are 4 ways of wearing it. Similarly, each of the other rings can be worn in 4 ways.

So, the required number of ways $= (4 \times 4 \times 4 \times 4 \times 4 \times 4) = 4^6$.

24. Since each letter can be posted in anyone of the 4 letter boxes, so each letter can be posted in 4 ways.

So, the required number of ways $= (4 \times 4 \times 4 \times 4 \times 4) = 4^5$.

25. (i) Total number of 3-letter words is equal to the number of ways of filling 3 places. First place can be filled in 5 ways by any of the given five letters. Second place can be filled in 4 ways by any of the remaining 4 letters and the third place can be filled in 3 ways by any of the remaining 3 letters.

So, the required number of 3-letter words $= (5 \times 4 \times 3) = 60$.

(ii) When repetition of letters is allowed, each place can be filled by any of the 5 letters in 5 ways.

$\therefore$ the required number of ways $= (5 \times 5 \times 5) = 125$.

26. In a 4-digit number, 0 cannot be placed at the thousand's place. So, thousand's place can be filled by any digit from 1 to 9 in nine ways. Now, each of hundred's, ten's and one's places can be filled by any of the digits from 0 to 9 in ten ways.

So, the required number of numbers $= (9 \times 10 \times 10 \times 10) = 9000$.

27. **One-digit numbers:**

Clearly, there are four 1-digit numbers.

Two-digit numbers:

We may fill the unit's place by any of the four given digits.

Thus, there are 4 ways to fill the unit's place.

The ten's place may now be filled by any of the remaining three digits. So, there are 3 ways to fill the ten's place.

Number of 2-digit numbers $= (4 \times 3) = 12$.

Three-digit numbers:

Number of ways to fill the unit's, ten's and hundred's places are 4, 3 and 2 respectively.

Number of 3-digit numbers $= (4 \times 3 \times 2) = 24$.

Four-digit numbers:

Number of ways to fill the unit's, ten's, hundred's and thousand's places are 4, 3, 2 and 1 respectively.

Number of 4-digit numbers $= (4 \times 3 \times 2 \times 1) = 24$.

Hence, the number of required numbers $= (4 + 12 + 24 + 24) = 64$.

28. The hundred's place cannot contain 0. So, it can be filled by any digit from 1 to 9. Thus, there are 9 ways to fill this place. Now, the ten's digit can be filled by any of the remaining 9 digits out of 0 to 9. So, there are 9 ways to fill the ten's place. And, there are 8 ways to fill the unit's digit.

Required number of numbers $= (9 \times 9 \times 8) = 648$.

29. The hundred's place cannot contain 0. So, it can be filled by any of the digits 1, 3, 5, 7. So, there are 4 ways to fill it.

Now, each of the ten's and unit's digits can be filled by any of the given five digits. So, each can be filled in 5 ways.

$\therefore$ the required number of numbers $= (4 \times 5 \times 5) = 100$.

30. The lakh's place cannot be 0. So, it can be filled by any of the digits 1, 3, 5, 7, 9. So, there are 5 ways to fill it. The ten-thousand's place can be filled by 0 or the remaining four digits, i,e., in 5 ways. The thousand's, hundred's, ten's and unit's digits can be filled in 4, 3, 2 and 1 ways respectively.

Required number of numbers $= (5 \times 5 \times 4 \times 3 \times 2 \times 1) = 600$.

Particular case Let us put 0 at unit's place, i.e., it is filled in 1 way.

Now ten's, hundred's, thousand's, ten-thousand's and lakh's place can be filled in 5, 4, 3, 2 and 1 ways respectively.

So, the number of numbers having 0 as unit's digit $= (5 \times 4 \times 3 \times 2 \times 1) = 120$.

So, 120 numbers are divisible by 10.

31. There are 5 one-digit natural numbers out of the given digits.

For getting 2-digit natural numbers we cannot put 0 at the ten's place. So, this place can be filled by any of the given five nonzero digits in 5 ways.

The unit's digit can be filled by any of the given six digits in 6 ways.

So, the number of 2-digit natural numbers $= (5 \times 6) = 30$.

Similarly, to get a 3-digit number, we cannot put 0 at the hundred's place. So, this place can be filled in 5 ways, each of the ten's and unit's places can be filled in 6 ways.

$\therefore$ the number of 3-digit numbers $= (5 \times 6 \times 6) = 180$.

Required number of numbers $= (5 + 30 + 180) = 215$.

32. Out of 10 digits from 0 to 9, we have used 2 digits, namely 6 and 7.

So, the 3rd, 4th, 5th and 6th digit may be filled in $(10 - 2) = 8, 7, 6$ and 5 ways respectively.

Hence, the required number of numbers $= (8 \times 7 \times 6 \times 5) = 1680$.

33. A can be assigned any of the three jobs. So, there are 3 ways of assigning a job to A. B can now be assigned any of the two remaining jobs in 2 ways and C can now be assigned the job in 1 way.

So, the required number of ways $= (3 \times 2 \times 1) = 6$.

34. The first term of the sequence may be anyone of the digits from 0 to 9. So, there are 10 ways of getting the first term.

So, the 2nd and 3rd terms may be obtained in 9 ways and 8 ways respectively.

Required number of sequences $= (10 \times 9 \times 8) = 720$.

Number of unsuccessful attempts $= (720 - 1) = 719$.

35. 1st, 2nd, 3rd, 4th digits may be chosen in 4 ways, 3 ways, 2 ways and 1 way respectively.

Number of largest possible trials $= (4 \times 3 \times 2 \times 1) = 24$.

36. (i) The 1st prize can be given to anyone of the 4 girls in 4 ways.

Similarly, the second prize can be given in 3 ways and the 3rd prize can be given in 2 ways.

$\therefore$ the required number of ways $= (4 \times 3 \times 2) = 24$.

(ii) 1st prize can be given in 4 ways, 2nd prize can be given in 4 ways and the 3rd prize can be given in 4 ways.

$\therefore$ the required number of ways $= (4 \times 4 \times 4) = 64$.

(iii) Number of ways in which a girl does not get all the prizes

= (number of ways in which a girl may get any number of prizes)

– (number of ways in which a girl gets all the prizes)

$= (64 - 4) = 60$.

PERMUTATIONS

PERMUTATIONS *The different arrangements which can be made out of a given number of things by taking some or all at a time, are called permutations.*

REMARK In permutations, the order of arrangement of the objects is taken into consideration. When the order is changed, a different permutation is obtained.

ILLUSTRATIVE EXAMPLES

EXAMPLE 1 *All permutations of the letters A, B, C taking two at a time are:*

AB, BA, AC, CA, BC, CB.

Thus, we obtain 6 different permutations.

EXAMPLE 2 *All permutations of the letters A, B, C taking all at a time are:*

ABC, ACB, BAC, BCA, CAB, CBA.

Thus, the 3 letters *A, B, C* taking all at a time give 6 permutations.

EXAMPLE 3 *All permutations of the letters A, B, C, D taking 3 at a time are:*

ABC, ACB, BAC, BCA, CAB, CBA;

ABD, ADB, BAD, BDA, CAD, CDA;

BCD, BDC, CBD, CDB, DBC, DCB;

ACD, ADC, CAD, CDA, DAC, DCA.

Thus, the 4 letters *A, B, C, D* taking 3 at a time give 24 permutations.

PERMUTATIONS WHEN ALL THE OBJECTS ARE DISTINCT

THEOREM 1 *The number of permutations of n different objects taken r at a time, where* $0 < r \leq n$ *and the objects do not repeat, is given by*

$$P(n, r) \text{ or } {}^{n}P_{r} = n(n-1)(n-2)\ldots(n-r+1).$$

PROOF The number of permutations of *n* different objects taken *r* at a time is the same as the number of ways of filling in *r* vacant places by the *n* objects.

The first place can be filled up by anyone of these *n* objects.

Thus, there are *n* ways of filling up the first place.

Now, we are left with $(n-1)$ objects.

The second place can be filled up by anyone of these $(n-1)$ objects.

Thus, there are $(n-1)$ ways of filling up the second place.

Now, we are left with $(n-2)$ objects.

The third place can be filled up by any of these $(n-2)$ objects.

Thus, there are $(n-2)$ ways of filling up the third place.

Similarly, the 4th place can be filled in $(n-3)$ ways, the 5th place can be filled in $(n-4)$ ways and so on, and the *r*th place can be filled in $\{n-(r-1)\} = (n-r+1)$ ways.

Thus, the total number of ways of filling in r vacant places by n different objects in succession is $\{n(n-1)(n-2)\ldots(n-r+1)\}$.

Hence, ${}^nP_r = n(n-1)(n-2)\ldots(n-r+1)$.

REMARK We have

$$\begin{aligned}{}^nP_r &= n(n-1)(n-2)\ldots(n-r+1)\\ &= \frac{n(n-1)(n-2)\ldots(n-r+1)(n-r)(n-r-1)\ldots3\cdot2\cdot1}{(n-r)(n-r-1)\ldots3\cdot2\cdot1}\\ &= \frac{n!}{(n-r)!}, \text{ where } 0 < r \le n.\end{aligned}$$

THEOREM 2 *The number of all permutations of n different objects taken all at a time is given by* ${}^nP_n = n!$.

PROOF The number of all permutations of n different objects taken all at a time is the same as the number of ways of filling n vacant places by n different objects.

Proceeding as in Theorem 1, we have

$${}^nP_n = n(n-1)(n-2)(n-3)\ldots\times3\times2\times1 = n!.$$

THEOREM 3 *Prove that* $0! = 1$.

PROOF We have

$${}^nP_r = \frac{n!}{(n-r)!} \qquad \ldots \text{(i)}$$

$$\Rightarrow \quad {}^nP_n = \frac{n!}{0!} \qquad [\text{putting } r = n \text{ in (i)}]$$

$$\Rightarrow \quad n! = \frac{n!}{0!} \qquad [\because\ {}^nP_n = n!]$$

$$\Rightarrow \quad 0! = \frac{n!}{n!} = 1.$$

Hence, $0! = 1$.

REMARK Thus, we have ${}^nP_r = \frac{n!}{(n-r)!}$, where $0 \le r \le n$.

SOLVED EXAMPLES

EXAMPLE 1 *Evaluate:*

(i) ${}^{12}P_4$ *(ii)* ${}^{75}P_2$ *(iii)* ${}^8P_8$

SOLUTION We have

$$\text{(i)}\ {}^{12}P_4 = \frac{12!}{(12-4)!} = \frac{12!}{8!} = \frac{12\times11\times10\times9\times(8!)}{8!} = 11880.$$

$$\text{(ii)}\ {}^{75}P_2 = \frac{75!}{(75-2)!} = \frac{75!}{73!} = \frac{75\times74\times(73!)}{73!} = (75\times74) = 5550.$$

$$\text{(iii)}\ {}^8P_8 = 8! = (8\times7\times6\times5\times4\times3\times2\times1) = 40320.$$

EXAMPLE 2 *Find the value of n such that*

(i) ${}^nP_5 = 42 \times {}^nP_3,\ n > 4$ *(ii)* $\dfrac{{}^nP_4}{{}^{n-1}P_4} = \dfrac{5}{3},\ n > 4.$

SOLUTION We know that ${}^nP_r = n(n-1)(n-2)\ldots(n-r+1)$.

(i) We have

$$\frac{{}^nP_5}{{}^nP_3} = 42 \Rightarrow \frac{n(n-1)(n-2)(n-3)(n-4)}{n(n-1)(n-2)} = 42$$

$$\Rightarrow (n-3)(n-4) = 42 \qquad [\text{as } n(n-1)(n-2) \neq 0]$$

$$\Rightarrow n^2 - 7n - 30 = 0 \Rightarrow (n-10)(n+3) = 0$$

$$\Rightarrow n = 10 \qquad [\because\ n \neq -3, \text{ as } n \text{ cannot be negative}].$$

Hence, $n = 10$.

(ii) $\dfrac{{}^nP_4}{{}^{n-1}P_4} = \dfrac{5}{3} \Rightarrow \dfrac{n(n-1)(n-2)(n-3)}{(n-1)(n-2)(n-3)(n-4)} = \dfrac{5}{3}$

$$\Rightarrow 3n = 5(n-4) \qquad [\text{as } (n-1)(n-2)(n-3)(n-4) \neq 0]$$

$$\Rightarrow 2n = 20 \Rightarrow n = 10.$$

Hence, $n = 10$.

EXAMPLE 3 *If* $5 \times {}^4P_r = 6 \times {}^5P_{r-1}$, *find r.*

SOLUTION We have

$$5 \times {}^4P_r = 6 \times {}^5P_{r-1}$$

$$\Rightarrow \quad 5 \times \frac{4!}{(4-r)!} = 6 \times \frac{5!}{\{5-(r-1)\}!}$$

$$\Rightarrow \quad \frac{5!}{(4-r)!} = \frac{6 \times (5!)}{(6-r)!} \qquad [\because\ 5 \times (4!) = 5!]$$

$$\Rightarrow \quad \frac{1}{(4-r)!} = \frac{6}{(6-r)!} \Rightarrow \frac{1}{(4-r)!} = \frac{6}{(6-r)(5-r)\{(4-r)!\}}$$

$$\Rightarrow \quad \frac{6}{(6-r)(5-r)} = 1 \Rightarrow (6-r)(5-r) - 6 = 0$$

$$\Rightarrow \quad r^2 - 11r + 24 = 0 \Rightarrow (r-3)(r-8) = 0$$

$$\Rightarrow \quad r = 3 \qquad [\because\ r \neq 8, \text{ as } {}^4P_8 \text{ is not defined}].$$

Hence, $r = 3$.

EXAMPLE 4 *If* ${}^nP_4 = 2 \times {}^5P_3$, *find n.*

SOLUTION We have

$${}^nP_4 = 2 \times {}^5P_3$$

$$\Rightarrow \quad n(n-1)(n-2)(n-3) = 2 \times 5 \times 4 \times 3$$

$$\Rightarrow \quad n(n-3)(n-1)(n-2) = 120$$

$$\Rightarrow \quad (n^2 - 3n)(n^2 - 3n + 2) = 120$$

$$\Rightarrow \quad m(m+2) - 120 = 0, \text{ where } m = n^2 - 3n$$

$$\Rightarrow \quad m^2 + 2m - 120 = 0 \Rightarrow (m+12)(m-10) = 0$$

$$\Rightarrow \quad m = -12 \text{ or } m = 10.$$

This gives

$$n^2 - 3n = -12 \quad \text{or} \quad n^2 - 3n = 10$$

$$\Rightarrow \quad n^2 - 3n + 12 = 0 \quad \text{or} \quad n^2 - 3n - 10 = 0$$

$$\Rightarrow \quad n = \frac{3 \pm \sqrt{9 - 48}}{2} \quad \text{or} \quad (n-5)(n+2) = 0$$

$$\Rightarrow \quad n = \frac{3 \pm i\sqrt{39}}{2} \quad \text{or} \quad n = 5 \ \text{or} \ n = -2$$

$\Rightarrow \quad n = 5$ [neglecting the negative and imaginary values of n].

Hence, $n = 5$.

EXAMPLE 5 *(i) If* ${}^{15}P_r = 2730$, *find the value of r.*

(ii) If ${}^{10}P_r = 5040$, *find the value of r.*

SOLUTION We have

(i) ${}^{15}P_r = 2730$

$\Rightarrow \quad {}^{15}P_r = 15 \times 182$

$\Rightarrow \quad {}^{15}P_r = 15 \times 14 \times 13$ (up to 3 factors)

$\Rightarrow \quad r = 3.$

Hence, $r = 3$.

(ii) ${}^{10}P_r = 5040$

$\Rightarrow \quad {}^{10}P_r = 10 \times 504$

$\Rightarrow \quad {}^{10}P_r = 10 \times 9 \times 56$

$\Rightarrow \quad {}^{10}P_r = 10 \times 9 \times 8 \times 7$ (up to 4 factors)

$\Rightarrow \quad r = 4.$

Hence, $r = 4$.

EXAMPLE 6 *If* ${}^{56}P_{r+6} : {}^{54}P_{r+3} = (30800 : 1)$, *find r.*

SOLUTION We have

$$^{56}P_{r+6} = \frac{56!}{[56-(r+6)]!} = \frac{56!}{(50-r)!}.$$

And, $$^{54}P_{r+3} = \frac{54!}{[54-(r+3)]!} = \frac{54!}{(51-r)!}.$$

$$\therefore \quad \frac{^{56}P_{r+6}}{^{54}P_{r+3}} = \frac{56!}{(50-r)!} \times \frac{(51-r)!}{54!}$$

$$= \frac{56 \times 55 \times (54!)}{(50-r)!} \times \frac{(51-r)\cdot[(50-r)!]}{54!}$$

$$= 56 \times 55 \times (51-r).$$

Thus, $\dfrac{^{56}P_{r+6}}{^{54}P_{r+3}} = \dfrac{30800}{1} \Rightarrow 56 \times 55 \times (51-r) = 30800$

$$\Rightarrow (51-r) = 10 \Rightarrow r = 41.$$

Hence, $r = 41$.

EXAMPLE 7 *If* ${}^{2n+1}P_{n-1} : {}^{2n-1}P_n = 3:5$, *find* n.

SOLUTION We have

$$
\begin{aligned}
& {}^{2n+1}P_{n-1} : {}^{2n-1}P_n = 3:5 \\
\Rightarrow \quad & \frac{(2n+1)!}{\{(2n+1)-(n-1)\}!} : \frac{(2n-1)!}{\{(2n-1)-n\}!} = \frac{3}{5} \\
\Rightarrow \quad & \frac{(2n+1)!}{(n+2)!} : \frac{(2n-1)!}{(n-1)!} = \frac{3}{5} \\
\Rightarrow \quad & \frac{(2n+1)!}{(n+2)!} \times \frac{(n-1)!}{(2n-1)!} = \frac{3}{5} \\
\Rightarrow \quad & \frac{(2n+1)\times 2n\times[(2n-1)!]}{(n+2)\times(n+1)\times n\times[(n-1)!]} \times \frac{(n-1)!}{(2n-1)!} = \frac{3}{5} \\
\Rightarrow \quad & 10(2n+1) = 3(n+2)(n+1) \\
\Rightarrow \quad & 20n+10 = 3(n^2+3n+2) \\
\Rightarrow \quad & 3n^2-11n-4 = 0 \\
\Rightarrow \quad & (n-4)(3n+1) = 0 \Rightarrow n = 4 \ \left[\because n \neq \frac{-1}{3}, \text{as } n \text{ cannot be negative}\right].
\end{aligned}
$$

Hence, $n = 4$.

EXAMPLE 8 *If* ${}^{22}P_{r+1} : {}^{20}P_{r+2} = 11:52$, *find* r.

SOLUTION We have

$$
\begin{aligned}
& {}^{22}P_{r+1} : {}^{20}P_{r+2} = 11:52 \\
\Rightarrow \quad & \frac{22!}{\{22-(r+1)\}!} : \frac{20!}{\{20-(r+2)\}!} = 11:52 \\
\Rightarrow \quad & \frac{22!}{(21-r)!} : \frac{20!}{(18-r)!} = \frac{11}{52} \\
\Rightarrow \quad & \frac{22!}{(21-r)!} \times \frac{(18-r)!}{20!} = \frac{11}{52} \\
\Rightarrow \quad & \frac{22\times 21\times(20!)}{(21-r)(20-r)(19-r)\times[(18-r)!]} \times \frac{(18-r)!}{20!} = \frac{11}{52} \\
\Rightarrow \quad & \frac{22\times 21}{(21-r)(20-r)(19-r)} = \frac{11}{52} \\
\Rightarrow \quad & (19-r)(20-r)(21-r) = 2\times 21\times 52 = 2\times 3\times 7\times 4\times 13 \\
\Rightarrow \quad & (19-r)(20-r)(21-r) = 12\times 13\times 14 \\
\Rightarrow \quad & 19-r = 12 \Rightarrow r = (19-12) = 7.
\end{aligned}
$$

Hence, $r = 7$.

EXAMPLE 9 *Prove that:*

(i) ${}^nP_n = {}^nP_{n-1}$ *(ii)* ${}^nP_r = n \cdot {}^{n-1}P_{r-1}$ *(iii)* ${}^{n-1}P_r + r \cdot {}^{n-1}P_{r-1} = {}^nP_r$

SOLUTION By using the formula for ${}^nP_r$, we have

(i) $${}^nP_{n-1} = \frac{n!}{\{n-(n-1)\}!} = \frac{n!}{1!} = n! = {}^nP_n.$$

(ii) $^nP_r = \dfrac{n!}{(n-r)!} = \dfrac{n\cdot(n-1)!}{[(n-1)-(r-1)]!} = n\cdot {}^{n-1}P_{r-1}.$

(iii) $^{n-1}P_r + r\cdot {}^{n-1}P_{r-1}$

$$= \left\{\frac{(n-1)!}{(n-1-r)!} + r\cdot\frac{(n-1)!}{[(n-1)-(r-1)]!}\right\}$$

$$= \left\{\frac{(n-1)!}{(n-1-r)!} + r\cdot\frac{(n-1)!}{(n-r)!}\right\}$$

$$= \left\{\frac{(n-1)!}{(n-r-1)!} + r\cdot\frac{(n-1)!}{(n-r)\cdot[(n-r-1)!]}\right\}$$

$$= \frac{(n-1)!}{(n-r-1)!}\left\{1+\frac{r}{(n-r)}\right\} = \frac{n\cdot[(n-1)!]}{(n-r)\cdot[(n-r-1)!]}$$

$$= \frac{n!}{(n-r)!} = {}^nP_r.$$

Hence, $^{n-1}P_r + r\cdot {}^{n-1}P_{r-1} = {}^nP_r.$

EXAMPLE 10 *If $r < s \le n$ then prove that ^nP_s is divisible by ^nP_r.*

SOLUTION We have

$$^nP_s = n(n-1)(n-2)\ldots(n-r+1)(n-r)\ldots(n-s+1)$$

$$= \{n(n-1)(n-2)\ldots(n-r+1)\}\times\{(n-r)(n-r-1)\ldots(n-s+1)\}$$

$$= {}^nP_r \times \{(n-r)(n-r-1)\ldots(n-s+1)\},$$

which is clearly divisible by ^nP_r.

Hence, ^nP_s is divisible by ^nP_r.

EXAMPLE 11 *Find the number of permutations of 7 objects, taken 3 at a time.*

SOLUTION Number of permutations of 7 objects, taken 3 at a time

$$= {}^7P_3 = 7\times6\times5 \quad \text{[up to three terms]}$$

$$= 210.$$

EXERCISE 8C

1. Evaluate:

(i) $^{10}P_4$ (ii) $^{62}P_3$ (iii) $^6P_6$ (iv) $^9P_0$

2. Prove that $^9P_3 + 3\times {}^9P_2 = {}^{10}P_3.$

3. (i) If $^nP_5 = 20\times {}^nP_3$, find n.

(ii) If $16\times {}^nP_3 = 13\times {}^{n+1}P_3$, find n.

(iii) If $^{2n}P_3 = 100\times {}^nP_2$, find n.

4. (i) If $^5P_r = 2\times {}^6P_{r-1}$, find r.

(ii) If $^{20}P_r = 13\times {}^{20}P_{r-1}$, find r.

(iii) If $^{11}P_r = {}^{12}P_{r-1}$, find r.

5. (i) If $^nP_4 : {}^nP_5 = 1:2$, find n.

(ii) If $^{n-1}P_3 : {}^{n+1}P_3 = 5:12$, find n.

6. If ${}^{15}P_{r-1} : {}^{16}P_{r-2} = 3:4$, find r.

7. If ${}^{2n-1}P_n : {}^{2n+1}P_{n-1} = 22:7$, find n.

8. Find n, if ${}^{n+5}P_{n+1} = \frac{11}{2}(n-1)\cdot {}^{n+3}P_n$.

9. Prove that $1 + 1\cdot {}^1P_1 + 2\cdot {}^2P_2 + 3\cdot {}^3P_3 + \ldots + n\cdot {}^nP_n = {}^{n+1}P_{n+1}$.

10. Find the number of permutations of 10 objects, taken 4 at a time.

ANSWERS (EXERCISE 8C)

1. (i) 5040 (ii) 226920 (iii) 720 (iv) 1 **3.** (i) $n = 8$ (ii) $n = 15$ (iii) $n = 13$

4. (i) $r = 3$ (ii) $r = 8$ (iii) $r = 9$ **5.** (i) $n = 6$ (ii) $n = 8$

6. $r = 14$ **7.** $n = 10$ **8.** $n = 6$ or $n = 7$ **10.** 5040

HINTS TO SOME SELECTED QUESTIONS

5. (ii) $\frac{(n-1)(n-2)(n-3)}{(n+1)n(n-1)} = \frac{5}{12} \Rightarrow 12(n-2)(n-3) = 5(n+1)n$

$\Rightarrow 7n^2 - 65n + 72 = 0 \Rightarrow 7n^2 - 56n - 9n + 72 = 0$

$\Rightarrow 7n(n-8) - 9(n-8) = 0 \Rightarrow (n-8)(7n-9) = 0 \Rightarrow n = 8.$

6. $\frac{15!}{\{15-(r-1)\}!} : \frac{16!}{\{16-(r-2)\}!} = 3:4 \Rightarrow \frac{15!}{(16-r)!} : \frac{16!}{(18-r)!} = 3:4.$

$\therefore \quad \frac{15!}{(16-r)!} \times \frac{(18-r)!}{16!} = \frac{3}{4} \Rightarrow \frac{15!}{(16-r)!} \times \frac{(18-r)(17-r)\cdot[(16-r)!]}{16\times(15!)} = \frac{3}{4}.$

$\therefore \quad (18-r)(17-r) = 12 \Rightarrow r^2 - 35r + 294 = 0 \Rightarrow (r-14)(r-21) = 0 \Rightarrow r = 14$

$[\because \ r \neq 21].$

7. $\frac{(2n-1)!}{(2n-1-n)!} : \frac{(2n+1)!}{\{(2n+1)-(n-1)\}!} = \frac{22}{7}$

$\Rightarrow \quad \frac{(2n-1)!}{(n-1)!} \times \frac{(n+2)!}{(2n+1)!} = \frac{22}{7} \Rightarrow \frac{(2n-1)!}{(n-1)!} \times \frac{(n+2)(n+1)n\cdot[(n-1)!]}{(2n+1)(2n)\cdot[(2n-1)!]} = \frac{22}{7}$

$\Rightarrow \quad 7(n+2)(n+1) = 44(2n+1) \Rightarrow 7n^2 - 67n - 30 = 0 \Rightarrow (n-10)(7n+3) = 0.$

Hence, $n = 10$.

9. $1 + 1\cdot(1!) + 2\cdot(2!) + 3\cdot(3!) + \ldots n\cdot(n!)$

$= 1 + (2-1)(1!) + (3-1)\cdot(2!) + (4-1)\cdot(3!) + \ldots + \{(n+1)-1\}\cdot(n!)$

$= \{1 + 2\cdot(1!) + 3\cdot(2!) + 4\cdot(3!) + \ldots + (n+1)\cdot(n!)\} - \{1! + 2! + 3! + \ldots + n!\}$

$= \{1 + 2! + 3! + 4! + \ldots + n! + (n+1)!\} - \{1! + 2! + 3! + \ldots + n!\} = (n+1)!.$

WORD PROBLEMS ON PERMUTATIONS

EXAMPLE 1 *In how many ways can* 6 *persons stand in a queue?*

SOLUTION Required number of ways

= number of arrangements of 6 different things taking all at a time

$= {}^6P_6 = 6! = (6\times5\times4\times3\times2\times1) = 720.$

EXAMPLE 2 *How many different signals can be made by 4 flags from 6 flags of different colours?*

SOLUTION Required number of signals

= number of arrangements of 6 flags taking 4 at a time

$= {}^6P_4 = (6 \times 5 \times 4 \times 3) = 360.$

EXAMPLE 3 *Eight athletes are participating in a race. In how many ways can the first 3 prizes be won by them?*

SOLUTION Total number of ways of winning first 3 prizes by 8 athletes

= number of arrangements of 8 athletes taking 3 at a time

$= {}^8P_3 = (8 \times 7 \times 6) = 336.$

EXAMPLE 4 *In how many ways can 3 different rings be worn in 4 fingers with at most one in each finger?*

SOLUTION Required number of ways

= number of ways of arranging 4 fingers taking 3 at a time

$= {}^4P_3 = (4 \times 3 \times 2) = 24.$

EXAMPLE 5 *3 men have 4 coats, 6 waistcoats and 5 caps. In how many ways can they wear them?*

SOLUTION Total number of ways in which 3 men can wear 4 coats

= number of arrangements of 4 coats taking 3 at a time

$= {}^4P_3 = (4 \times 3 \times 2) = 24.$

Total number of ways in which 3 men can wear 6 waistcoats

= number of arrangements of 6 waistcoats taking 3 at a time

$= {}^6P_3 = (6 \times 5 \times 4) = 120.$

Total number of ways in which 3 men can wear 5 caps

$= {}^5P_3 = (5 \times 4 \times 3) = 60.$

Hence, by the fundamental principle of multiplication, the required number of ways $= (24 \times 120 \times 60) = 172800.$

EXAMPLE 6 *How many different signals can be given by using any number of flags from 4 flags of different colours?*

SOLUTION These signals can be made by using 1 or 2 or 3 or 4 flags at a time.

Number of signals made by 4 flags:

(i) taking 1 at a time $= {}^4P_1 = 4;$

(ii) taking 2 at a time $= {}^4P_2 = (4 \times 3) = 12;$

(iii) taking 3 at a time $= {}^4P_3 = (4 \times 3 \times 2) = 24;$

(iv) taking 4 at a time $= {}^4P_4 = (4 \times 3 \times 2 \times 1) = 24.$

Hence, by the fundamental principle of addition, the total number of signals $= (4 + 12 + 24 + 24) = 64.$

EXAMPLE 7 *A code is to consist of two distinct letters followed by two distinct numbers between* 1 *and* 9. *For example, CA* 35, *DQ* 72, *etc., are codes.*

(i) How many such codes are there?

(ii) How many of them end with an even integer?

SOLUTION (i) We know that there are 26 letters in the English alphabet.

Number of ways of choosing 2 letters out of 26

$$= {}^{26}P_2 = (26 \times 25) = 650.$$

Number of ways of choosing 2 numbers out of 9

$$= {}^{9}P_2 = (9 \times 8) = 72.$$

By the fundamental principle of multiplication, the total number of codes = $(650 \times 72) = 46800$.

(ii) **Particular case** *When each code ends with an even integer*

In this case, it can have any of the numbers 2, 4, 6, 8 at the extreme right position. This position can thus be filled in 4 ways.

The next left position can be filled with any of the remaining 8 numbers. So, there are 8 ways of filling this position.

Number of ways of choosing pairs of even numbers

$$= (4 \times 8) = 32.$$

Number of ways of choosing 2 letters out of 26

$$= {}^{26}P_2 = (26 \times 25) = 650.$$

Hence, the number of such codes which end with an even integer = $(32 \times 650) = 20800$.

EXAMPLE 8 *Find the number of ways in which* 5 *boys and* 3 *girls can be arranged in a row so that no two girls are together.*

SOLUTION In order that no two girls are together, we must arrange the 5 boys, each denoted by *B*, as under:

$$X\ B\ X\ B\ X\ B\ X\ B\ X\ B\ X$$

Here, *B* denotes the position of a boy and *X* that of a girl.

Number of ways in which 5 boys can be arranged at 5 places

$$= {}^{5}P_5 = 5! = (5 \times 4 \times 3 \times 2 \times 1) = 120.$$

Number of ways in which 3 girls can be arranged at 6 places (each marked *X*) = ${}^{6}P_3 = (6 \times 5 \times 4) = 120$.

Hence, by the fundamental principle of multiplication, the required number of ways = $(120 \times 120) = 14400$.

EXAMPLE 9 *There are* 2 *English,* 3 *Sanskrit and* 4 *Hindi books. In how many ways can they be arranged on a shelf so as to keep all the books of the same language together?*

SOLUTION Let us make 1 packet for each of the books on the same language.

Number of ways of arranging the 3 packets = ${}^{3}P_3 = 3! = 6$.

Number of ways of arranging 2 English books $= 2! = 2$.

Number of ways of arranging 3 Sanskrit books $= 3! = 6$.

Number of ways of arranging 4 Hindi books $= 4! = 24$.

Hence, the required number of ways $= (2 \times 6 \times 24) \times 6 = 1728$.

EXAMPLE 10 *In how many ways can 7 examination papers be arranged so that the best and the worst papers are never together?*

SOLUTION Let us tie the best paper (b) and the worst paper (w) together and consider (bw) as one paper.

Now, this (bw) and 5 other papers may be arranged in

$${}^6P_6 = 6! = 720 \text{ ways.}$$

Also, these two papers may be arranged among themselves in $2! = 2$ ways.

Total number of arrangements with best and worst papers together

$$= (720 \times 2) = 1440.$$

Total number of ways of arranging 7 papers

$$= {}^7P_7 = 7! = 5040.$$

Number of arrangements with best and worst paper never together

$$= (5040 - 1440) = 3600.$$

EXAMPLE 11 *In how many ways can 6 balls of different colours, namely black, white, blue, red, green and yellow be arranged in a row in such a way that the black and white balls are never together?*

SOLUTION Let us tie the black ball (b) and white ball (w) together and consider (bw) as one ball.

Now, this (bw) and 4 other balls may be arranged in

$${}^5P_5 = 5! = 120 \text{ ways.}$$

Also, these two balls may be arranged among themselves in $2! = 2$ ways.

Total number of arrangements with black and white balls together

$$= (120 \times 2) = 240.$$

Number of ways of arranging 6 balls among themselves

$$= {}^6P_6 = 6! = 720.$$

Number of ways of arranging 6 balls such that black and white balls are never together $= (720 - 240) = 480$.

EXAMPLE 12 *The principal wants to arrange 5 students on a platform such that the boy Sajal occupies the second position and the girl Tanvy is always adjacent to the girl Aditi. How many such arrangements are possible?*

SOLUTION Let the five seats be arranged as shown below.

I	II	III	IV	V
=	S	=	=	=

Keep Sajal fixed at the second position.

Since Tanvy and Aditi are to sit together, none of them can occupy the first seat.

The first seat can be occupied by any of the two remaining students in $2! = 2$ ways.

Number of ways in which 2 seats namely III, IV or IV, V may be occupied by Tanvy and Aditi $= (2!) + (2!) = (2+2) = 4$.

The remaining seat may now be occupied by the 5th student in 1 way only.

Hence, the required number of arrangements $= (2 \times 4 \times 1) = 8$.

EXAMPLE 13 *How many words (with or without meaning) can be formed by using all the letters of the word, 'DELHI' using each letter exactly once?*

SOLUTION The word, 'DELHI' contains 5 different letters.

Total number of words formed by using all the letters of the given word $= {}^5P_5 = 5! = (5 \times 4 \times 3 \times 2 \times 1) = 120$.

Hence, the required number of words = 120.

EXAMPLE 14 *How many 4-letter words (with or without meaning) can be formed out of the letters of the word, 'LOGARITHMS', if repetition of letters is not allowed?*

SOLUTION The word, 'LOGARITHMS' contains 10 different letters.

Number of 4-letter words formed out of 10 given letters

$= {}^{10}P_4 = (10 \times 9 \times 8 \times 7) = 5040$.

Hence, the required number of 4-letter words = 5040.

EXAMPLE 15 *How many words (with or without meaning) can be formed from the letters of the word, 'DAUGHTER', so that*

(i) all vowels occur together?

(ii) all vowels do not occur together?

SOLUTION The given word, 'DAUGHTER' contains 3 vowels A, U, E and 5 consonants D, G, H, T, R.

Case (i) *When all vowels occur together:*

Let us assume (AUE) as a single letter.

Then, this letter (AUE) along with 5 other letters can be arranged in ${}^6P_6 = (6!)$ ways $= (6 \times 5 \times 4 \times 3 \times 2 \times 1)$ ways

$= 720$ ways.

These 3 vowels may be arranged among themselves in $3! = 6$ ways.

Hence, the required number of words with vowels together

$= (6!) \times (3!) = (720 \times 6) = 4320$.

Case (ii) *When all vowels do not occur together:*

Number of words formed by using all the 8 letters of the given word

$= {}^8P_8 = 8! = (8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1) = 40320.$

Number of words in which all vowels are never together

= (total number of words) – (number of words with all vowels together)

$= (40320 - 4320) = 36000.$

EXAMPLE 16 *How many words (with or without meaning) can be made from the letters of the word 'MONDAY', assuming that no letter is repeated, when:*

(i) 4 letters are used at a time?

(ii) all letters are used at a time?

(iii) all letters are used but the first letter is a vowel?

SOLUTION The given word 'MONDAY' contains 6 letters out of which two letters namely O and A are vowels.

Case (i) *When 4 letters are used at a time:*

Then, the required number of words

= number of arrangements of 6 letters taking 4 at a time

$= {}^6P_4 = (6 \times 5 \times 4 \times 3) = 360.$

Case (ii) *When all the letters are used at a time:*

Then, the required number of words

= number of arrangements of 6 letters taking all at a time

$= {}^6P_6 = 6! = (6 \times 5 \times 4 \times 3 \times 2 \times 1) = 720.$

Case (iii) *When all the letters are used and the first letter is a vowel:*

The given word contains 2 vowels, namely O and A.

So, the first letter can be chosen 1 out of 2 vowels.

Thus, there are 2 ways of choosing the first letter.

Number of ways of arranging the remaining 5 letters among themselves $= 5! = (5 \times 4 \times 3 \times 2 \times 1) = 120.$

Hence, the required number of words $= (2 \times 120) = 240.$

EXAMPLE 17 *(i) How many words can be formed from the letters of the word, 'TRIANGLE'?*

(ii) How many of them will begin with T and end with E?

SOLUTION (i) The given word 'TRIANGLE' contains 8 letters.

Total number of words formed with these 8 letters

$= {}^8P_8 = 8! = 40320.$

Hence, the required number of words = 40320.

(ii) By fixing *T* in the beginning and *E* at the end, the remaining 6 letters can be arranged among themselves in $^6P_6 = 6! = 720$ ways.

Hence, the total number of words, each beginning with *T* and ending with $E = 720$.

EXAMPLE 18 *How many words can be formed with the letters of the word 'ORDINATE' so that the vowels occupy odd places?*

SOLUTION The given word consists of 8 letters, out of which there are 4 vowels and 4 consonants.

Let us mark out the positions to be filled up, as shown below:

$$(^1)(^2)(^3)(^4)(^5)(^6)(^7)(^8).$$

Since vowels occupy odd places, they may be placed at 1, 3, 5, 7.

Number of ways of arranging 4 vowels at 4 odd places

$$= {}^4P_4 = 4! = 24.$$

The remaining 4 letters of the given word are consonants, which can be arranged at 4 even places marked 2, 4, 6, 8.

Number of ways of arranging 4 consonants at 4 even places

$$= {}^4P_4 = 4! = 24.$$

Hence, the total number of words in which the vowels occupy odd places $= (24 \times 24) = 576$.

EXAMPLE 19 *How many 3-digit numbers can be formed by using the digits 1 to 9, if no digit is repeated?*

SOLUTION Required number of numbers

= number of permutations of 9 digits taking 3 at a time

$= {}^9P_3 = (9 \times 8 \times 7) = 504.$

Hence, the required number of numbers = 504.

EXAMPLE 20 *Find the number of 4-digit numbers that can be formed using the digits 1, 2, 3, 4, 5 if no digit is repeated. How many of them will be even?*

SOLUTION Required number of 4-digit numbers

= number of arrangements of 5 digits taking 4 at a time

$= {}^5P_4 = (5 \times 4 \times 3 \times 2) = 120.$

Particular case To get an even number, the unit's digit is 2 or 4.

Thus, there are 2 ways of filling the unit's digit.

Number of ways of filling remaining 3 places with the remaining 4 digits $= {}^4P_3 = (4 \times 3 \times 2) = 24$.

Hence, the required number of even numbers $= (2 \times 24) = 48$.

EXAMPLE 21 *How many 6-digit telephone numbers can be constructed with the digits 0, 1, 2, 3, 4, 5, 6, 7, 8, 9 if each number starts with 35 and no digit appears more than once?*

SOLUTION Out of the given 10 digits, the two digits namely 3 and 5 are reserved for the two extreme left places of the desired numbers.

The remaining 8 digits can be arranged in the remaining four positions in ${}^8P_4 = (8 \times 7 \times 6 \times 5) = 1680$ ways.

Hence, 1680 telephone numbers can be constructed.

EXERCISE 8D

1. In how many ways can 5 persons occupy 3 vacant seats?
2. In how many ways can 7 people line up at a ticket window of a cinema hall?
3. In how many ways can 5 children stand in a queue?
4. In how many ways can 6 women draw water from 6 wells, if no well remains unused?
5. In how many ways can 4 different books, one each in chemistry, physics, biology and mathematics, be arranged on a shelf?
6. Six students are contesting the election for the presidentship of the students' union. In how many ways can their names be listed on the ballot papers?
7. It is required to seat 5 men and 3 women in a row so that the women occupy the even places. How many such arrangements are possible?
8. There are 6 items in column A and 6 items in column B. A student is asked to match each item in column A with an item in column B. How many possible, correct or incorrect answers are there to this question?
9. Five letters F, K, R, T and V one in each were purchased from a plastic warehouse. How many ordered pairs of letters, to be used as initials, can be formed from them?
10. Ten students are participating in a race. In how many ways can the first three prizes be won?
11. If there are 6 periods on each working day of a school, in how many ways can one arrange 5 subjects such that each subject is allowed at least one period?
12. In how many ways can 6 pictures be hung from 4 picture nails on a wall?
13. Find the number of words formed (may be meaningless) by using all the letters of the word 'EQUATION', using each letter exactly once.
14. Find the number of different 4-letter words (may be meaningless) that can be formed from the letters of the word 'NUMBERS'.
15. How many words can be formed from the letters of the word 'SUNDAY'? How many of these begin with D?
16. How many words beginning with C and ending with Y can be formed by using the letters of the word 'COURTESY'?
17. Find the number of permutations of the letters of the word 'ENGLISH'. How many of these begin with E and end with I?
18. In how many ways can the letters of the word 'HEXAGON' be permuted? In how many words will the vowels be together?
19. How many words can be formed out of the letters of the word 'ORIENTAL' so that the vowels always occupy the odd places?
20. In how many ways can the letters of the word 'FAILURE' be arranged so that the consonants may occupy only odd positions?
21. In how many arrangements of the word 'GOLDEN' will the vowels never occur together?

22. Find the number of ways in which the letters of the word 'MACHINE' can be arranged such that the vowels may occupy only odd positions.

23. How many permutations can be formed by the letters of the word 'VOWELS', when

(i) there is no restriction on letters;

(ii) each word begins with E;

(iii) each word begins with O and ends with L;

(iv) all vowels come together;

(v) all consonants come together?

24. How many numbers divisible by 5 and lying between 3000 and 4000 can be formed by using the digits 3, 4, 5, 6, 7, 8 when no digit is repeated in any such number?

25. In an examination, there are 8 candidates out of which 3 candidates have to appear in mathematics and the rest in different subjects. In how many ways can they be seated in a row, if candidates appearing in mathematics are not to sit together?

26. In how many ways can 5 children be arranged in a line such that

(i) two of them, Rajan and Tanvy, are always together?

(ii) two of them, Rajan and Tanvy, are never together?

27. When a group photograph is taken, all the seven teachers should be in the first row and all the twenty students should be in the second row. If the two corners of the second row are reserved for the two tallest students, interchangeable only between them, and if the middle seat of the front row is reserved for the principal, how many arrangements are possible?

28. Find the number of ways in which m boys and n girls may be arranged in a row so that no two of the girls are together, it being given that $m > n$.

ANSWERS (EXERCISE 8D)

1. 60	**2.** 5040	**3.** 120	**4.** 720	**5.** 24
6. 720	**7.** 7200	**8.** 720	**9.** 20	**10.** 720
11. 600	**12.** 360	**13.** 40320	**14.** 840	**15.** 720, 120
16. 720	**17.** 5040, 120	**18.** 5040, 720	**19.** 576	**20.** 576
21. 480	**22.** 576	**23.** (i) 720 (ii) 120 (iii) 24 (iv) 240 (v) 144		
24. 12	**25.** 14400	**26.** (i) 48 (ii) 72	**27.** $(18!)\times(6!)\times 2$	

28. $\dfrac{(m!)\times(m+1)!}{(m+1-n)!}$

HINTS TO SOME SELECTED QUESTIONS

7. 5 men can be arranged at 5 places in ${}^5P_5 = 5! = 120$ ways.

Starting from man, we leave alternate places for women, as shown below.

M = M = M = M = M =

Now, 3 women can be arranged at these 5 places, marked =, in

$${}^5P_3 = (5 \times 4 \times 3) = 60 \text{ ways.}$$

Required number of ways $= (120 \times 60) = 7200.$

9. Number of ordered pairs $= {}^5P_2 = (5 \times 4) = 20.$

10. Required number of ways $= {}^{10}P_3.$

11. Out of 6 periods, 5 may be arranged for 5 subjects in ${}^5P_5$ ways.

Remaining 1 period may be arranged for anyone of the five subjects in ${}^5P_1$ ways.

Required number of ways $= ({}^5P_5 \times {}^5P_1).$

18. The given word contains 7 letters, which may be arranged among themselves in $7! = 5040$ ways.

The given word contains 3 vowels and 4 consonants.

Taking the 3 vowels EAO as one letter, this letter and 4 more letters can be arranged in $5! = 120$ ways.

The 3 vowels can be arranged among themselves in $3! = 6$ ways.

Required number of arrangements with vowels together $= (120 \times 6) = 720.$

19. The given word contains 4 vowels and 4 consonants.

Let us fix the 8 places as shown below.

$$(1)\ (2)\ (3)\ (4)\ (5)\ (6)\ (7)\ (8)$$

The 4 vowels can be arranged at 4 places (namely 1, 3, 5, 7) in ${}^4P_4 = 4! = 24$ ways.

And, the 4 consonants can be arranged at 4 places (namely 2, 4, 6, 8) in ${}^4P_4 = 4!$ = 24 ways.

Required number of ways $= (24 \times 24) = 576.$

20. The given word 'FAILURE' has 3 consonants and 4 vowels, which can be arranged at seven places, shown below.

$$(1)\ (2)\ (3)\ (4)\ (5)\ (6)\ (7)$$

Now, 3 consonants may be placed at any of the 3 places out of the 4 marked 1, 3, 5, 7.

Number of ways of arranging the consonants $= {}^4P_3 = 24.$

And, the 4 vowels can be arranged at the remaining 4 places in ${}^4P_4 = 24$ ways.

Required number of ways $= (24 \times 24) = 576.$

21. The given word 'GOLDEN' contains 6 letters which may be arranged among themselves in $6! = 720$ ways.

It has 2 vowels and 4 consonants.

Taking the 2 vowels OE as a single letter, this letter and 4 consonants together may be arranged in $5! = 120$ ways.

Also, O and E may be arranged among themselves in 2! ways.

Number of arrangements with vowels together $= (120 \times 2) = 240.$

Number of arrangements with vowels never together $= (720 - 240) = 480.$

22. There are 7 letters in the given word, out of which there are 3 vowels and 4 consonants.

Let us mark out the positions to be filled up as follows:

$$(1)\ (2)\ (3)\ (4)\ (5)\ (6)\ (7)$$

Now, the 3 vowels can be placed at any of the three places out of the four, marked 1, 3, 5, 7.

So, the number of ways of arranging the vowels $= {}^4P_3 = 4 \times 3 \times 2 = 24.$

Also, the 4 consonants at the remaining 4 positions may be arranged in ${}^4P_4 = 4!$ = 24 ways.

$\therefore$ the requisite number of ways $= (24 \times 24) = 576$.

24. Fix up 3 at the thousand's place and 5 at the unit's place. Now, the remaining 2 places can be filled by the remaining 4 digits in ${}^4P_2 = (4 \times 3) = 12$ ways.

25. Clearly, there are 5 candidates not appearing in mathematics.

Let us arrange these 5 in a row, each shown by *X*.

They can be arranged in $5! = 120$ ways.

On both sides of each *X*, we put an *M*, as shown below:

$$MXMXMXMXMXM$$

Now, 3 candidates in mathematics can be arranged at 6 places in ${}^6P_3$ ways $= (6 \times 5 \times 4)$ ways = 120 ways.

Total number of arrangements $= (120 \times 120) = 14400$.

26. Total number of ways of arranging 5 children in a row $= 5! = 120$.

(i) Considering $\boxed{\text{Rajan + Tanvy}}$ as one child, this one and 3 more can be arranged in $4! = 24$ ways.

Rajan and Tanvy can be arranged among themselves in $2! = 2$ ways.

Required number of ways $= (24 \times 2) = 48$.

(ii) When Rajan and Tanvy are never together, the number of arrangements $= (120 - 48) = 72$.

27. Keeping the middle seat of the front row fixed for the principal, the remaining 6 seats can be occupied by the remaining 6 teachers in 6 ! ways.

The two corner seats may be occupied by the two tallest students in 2 ways. The remaining 18 seats may be occupied by the remaining 18 students in 18 ! ways.

28. We may arrange *m* boys at *m* places in *m* ! ways, leaving a seat between each pair of boys as shown below:

$$\times B \times B \times \ldots \times B \times .$$

Now, *n* girls may be arranged in $(m+1)$ places, shown by cross marks.

The number of such arrangements is ${}^{m+1}P_n$.

PERMUTATIONS OF OBJECTS NOT ALL DIFFERENT

THEOREM 1 *Let there be n objects, of which m objects are alike of one kind, and the remaining $(n-m)$ objects are alike of another kind. Then, the total number of mutually distinguishable permutations that can be formed from these objects is*

$$\frac{n!}{(m!) \times (n-m)!}.$$

PROOF Let the required number of permutations be *x*.

Consider any of these *x* permutations.

Now, replace *m* like things in this permutation by *m* different things. These *m* different things may be arranged amongst themselves in $m!$ ways.

Similarly, if we replace $(n-m)$ like things by $(n-m)$ different things, then they can be arranged amongst themselves in $(n-m)!$ ways.

Thus, if both the replacements are done simultaneously then each one of the *x* permutations gives rise to $(m!)\times(n-m)!$ permutations.

x permutations give rise to $x\times(m!)\times(n-m)!$ permutations.

Now, each of these $x\times(m!)\times(n-m)!$ permutations is a permutation of *n* different things, taken all at a time.

$\therefore \quad x\times(m!)\times(n-m)!=n!.$

Hence, $x=\dfrac{n!}{(m!)\times(n-m)!}.$

REMARK 1 *Let there be n things of which p_1 are alike of one kind, p_2 are alike of another kind, p_3 are alike of 3rd kind, …, p_r are alike of rth kind such that $p_1+p_2+\ldots+p_r=n$. Then, the number of permutations of these n things is*

$$\frac{n!}{(p_1!)\times(p_2!)\times\ldots(p_r!)}.$$

REMARK 2 *The number of permutations of n things of which p are alike of one kind, q are alike of another kind and remaining all are distinct is* $\dfrac{n!}{(p!)\times(q!)}$.

SOLVED EXAMPLES

EXAMPLE 1 *Find the number of different permutations of the letters of the word, 'BANANA'.*

SOLUTION The given word 'BANANA' contains 6 letters, out of which three are alike of one kind (3 A's), two are alike of second kind (2 N's) and one of its own kind (1 B).

Total number of their permutations $=\dfrac{6!}{(3!)\times(2!)\times(1!)}=60.$

EXAMPLE 2 *How many words can be formed using the letter A thrice, the letter B twice and the letter C thrice?*

SOLUTION We are given 8 letters, namely AAABBCCC. Out of these letters three are alike of one kind (3 A's), two are alike of second kind (2 B's) and three are alike of third kind (3 C's).

Hence, the required number of permutations

$$=\frac{8!}{(3!)\times(2!)\times(3!)}=560.$$

EXAMPLE 3 *(i) Find how many arrangements can be made with the letters of the word 'MATHEMATICS'.*

(ii) In how many of them are the vowels together?

SOLUTION (i) There are 11 letters in the word 'MATHEMATICS'. Out of these letters M occurs twice, A occurs twice, T occurs twice and the rest are all different.

Hence, the total number of arrangements of the given letters

$$= \frac{11!}{(2!)\times(2!)\times(2!)} = 4989600.$$

(ii) The given word contains 4 vowels, namely A, E, A, I. Treating these 4 vowels $\boxed{AEAI}$ as one letter, we have to arrange 8 letters MTHMTCS + $\boxed{AEAI}$, out of which M occurs twice, T occurs twice and the rest are all different.

So, the number of all such arrangements $= \frac{8!}{(2!)\times(2!)} = 10080.$

Now, out of 4 vowels, A occurs twice and the rest are all distinct.

So, the number of arrangements of these vowels $= \frac{4!}{2!} = 12.$

Hence, the number of arrangements in which 4 vowels are together $= (10080 \times 12) = 120960.$

EXAMPLE 4 *I. Find the number of arrangements of the letters of the word, 'INDEPENDENCE'.*

II. In how many of these arrangements:

(i) do the words start with P?

(ii) do all the vowels occur together?

(iii) do the vowels never occur together?

(iv) do the words begin with I and end in P?

SOLUTION I. The given word contains 12 letters out of which N occurs 3 times, E occurs 4 times, D occurs 2 times and the rest are all different.

Hence, the required number of arrangements

$$= \frac{12!}{(3!)\times(4!)\times(2!)} = 1663200.$$

II. (i) After fixing *P* at the extreme left position, there are 11 letters consisting of 3 N's, 4 E's, 2 D's, 1 I and 1 C.

So, the number of words beginning with *P*

= number of arrangements of 11 letters out of which there are 3 N's, 4 E's, 2 D's, 1 I and 1 C

$$= \frac{11!}{(3!)\times(4!)\times(2!)} = 138600.$$

(ii) There are 5 vowels in the given word, namely 4 E's and 1 I.

Let us treat them as a single letter $\boxed{EEEEI}$.

This letter with 7 remaining letters will give us 8 letters in which there are 3 N's, 2 D's, 1 P, 1 C and 1 letter $\boxed{EEEEI}$.

Number of all such arrangements $= \frac{8!}{(3!)\times(2!)} = 3360.$

Number of arrangements of 5 vowels, namely 4 E's and 1 I

$$= \frac{5!}{4!} = 5.$$

Hence, the number of words in which vowels are together

$$= (3360 \times 5) = 16800.$$

(iii) Number of arrangements in which vowels do not occur together

= (total number of arrangements) – (number of arrangements in which vowels occur together)

$= (1663200 - 16800) = 1646400.$

(iv) Let us fix I at the left end and P at the right end of each arrangement.

Then, we are left with 10 letters, out of which there are 3 N's, 4 E's, 2 D's and 1 C.

Hence, the number of words which begin with I and end in P $= \frac{10!}{(3!) \times (4!) \times (2!)} = 12600.$

EXAMPLE 5 *(i) How many different words can be formed by using all the letters of the word, 'ALLAHABAD'?*

In how many of them:

(ii) both L's do not come together?

(iii) the vowels occupy the even positions?

SOLUTION (i) The given word, 'ALLAHABAD' contains 9 letters consisting of 4 A's, 2 L's, 1 H, 1 B and 1 D.

Hence, the number of different words formed by using all the letters of the given word $= \frac{9!}{(4!) \times (2!)} = 7560.$

(ii) Let us take both L together and we treat $\boxed{LL}$ as 1 letter.

Then, we will have to arrange 8 letters, namely $\boxed{LL}$, 4 A's, 1 H, 1 B and 1 D.

So, the number of words having both L together $= \frac{8!}{4!} = 1680.$

Hence, the number of words with both L not occurring together

$= (7560 - 1680) = 5880.$

(iii) There are 4 vowels and all are alike, i.e., 4 A's.

Also, there are 4 even places, namely 2nd, 4th, 6th and 8th.

So, we arrange A's at these 4 places, as shown below.

$= A = A = A = A =$

Number of arrangements of 4 A's at 4 places $= \frac{4!}{4!} = 1.$

Now, we are left with 5 letters consisting of 2 L's, 1 H, 1 B and 1 D.

Number of arrangements of these 5 letters at 5 places $= \frac{5!}{2!} = 60.$

Hence, the number of words in which vowels occupy even places $= (1 \times 60) = 60.$

EXAMPLE 6 *In how many of the distinct permutations of the letters in 'MISSISSIPPI' do the 4 I's not come together?*

SOLUTION The given word 'MISSISSIPPI' contains 11 letters, out of which there are 4 I's, 4 S's, 2 P's and 1 M.

Total number of words formed by the letters of the given word

$$= \frac{11!}{(4!) \times (4!) \times (2!)} = 34650.$$

There are 4 I's in the given word and we treat $\boxed{\text{IIII}}$ as 1 letter.

This single letter with remaining 7 letters will give us 8 letters out of which there are 4 S's, 2 P's, 1 M and 1 $\boxed{\text{IIII}}$.

So, the number of all such arrangements in which 4 I's are together

$$= \frac{8!}{(4!) \times (2!)} = 840.$$

Thus, the number of words formed when 4 I's are together = 840.

Hence, the number of words formed when 4 I's are not together

$$= (34650 - 840) = 33810.$$

EXAMPLE 7 *(i) How many words can be formed with the letters of the word, 'HARYANA'?*

How many of these

(ii) have H and N together?

(iii) begin with H and end with N?

(iv) have 3 vowels together?

SOLUTION (i) The given word 'HARYANA' consists of 7 letters, out of which there are 1 H, 3 A's, 1 R, 1 Y and 1 N.

Total number of words formed by all the letters of the given word $= \frac{7!}{3!} = 840.$

(ii) Let us consider $\boxed{\text{HN}}$ as a single letter.

Now, $\boxed{\text{HN}}$ + ARYAA will give us 6 letters out of which there are 3 A's, 1 R, 1 Y and 1 $\boxed{\text{HN}}$.

Total number of all such arrangements $= \frac{6!}{3!} = 120.$

But, H and N can be arranged amongst themselves in 2! ways.

Hence, the number of words having H and N together

$$= (120 \times 2) = 240.$$

(iii) After fixing H in first place and N in last place, we have 5 letters, out of which there are 3 A's, 1 R and 1 Y.

Hence, the number of words beginning with H and ending with N $= \frac{5!}{3!} = 20.$

(iv) The given word contains 3 vowels AAA and let us treat $\boxed{\text{AAA}}$ as 1 letter.

Now, we have to arrange 5 letters HRYN + $\boxed{\text{AAA}}$ at 5 places.

Hence, total number of words formed having all vowels together $= 5! = (5 \times 4 \times 3 \times 2 \times 1) = 120.$

EXAMPLE 8 *In how many ways can the letters of the word 'PERMUTATIONS' be arranged, if*

(i) the words start with P and end with S?

(ii) the vowels are all together?

SOLUTION (i) Let us fix P at the left end and S at the right end of each arrangement.

Then, we are left with 10 letters, out of which T occurs 2 times and the rest are all different.

Hence, the required number of arrangements $= \frac{10!}{2!} = 1814400.$

(ii) The given word contains 5 vowels, namely E, U, A, I, O.

Treating these five vowels $\boxed{\text{EUAIO}}$ as one letter, we have to arrange 8 letters, namely $\boxed{\text{EUAIO}}$ + PRMTTNS, out of which T occurs 2 times and the rest are all different.

Number of all such arrangements $= \frac{8!}{2!} = 20160.$

Now, the 5 vowels are all different. They can be arranged among themselves in $5! = 120$ ways.

Hence, the number of arrangements in which 5 vowels are together $= (20160 \times 120) = 2419200.$

EXAMPLE 9 *If the different permutations of the word 'EXAMINATION' are listed as in a dictionary, how many items are there in the list before the first word starting with E?*

SOLUTION In a dictionary, the words at each stage are arranged in an alphabetical order.

When the letters of the word 'EXAMINATION' are arranged in the dictionary, the first word would be AAEIIMNNOTX.

Starting with A, we arrange the remaining 10 letters AEIIMNNOTX in which there are 2 I's, 2 N's and the rest are all different, in all possible ways.

Thus, the number of words formed beginning with A

$$= \frac{10!}{(2!) \times (2!)} = 907200.$$

In the dictionary, the next word would be EAAIIMNNOTX.

Hence, 907200 words are there in the dictionary before the first word starting with E.

EXAMPLE 10 *(i) Find the number of words which can be made using all the letters of the word, 'AGAIN'.*

(ii) If these words are written as in a dictionary, what will be the 50th word?

SOLUTION (i) The given word 'AGAIN' consists of 5 letters, out of which there are 2 A's and the rest are all distinct.

Hence, the number of words formed by using all the letters of the given word $= \frac{5!}{2!} = 60.$

(ii) When the letters of the word 'AGAIN' are listed in a dictionary, the first word would be AAGIN.

Starting with A and arranging the remaining 4 letters A, G, I, N, we obtain 4 ! words, i.e., 24 words.

The 25th word would be GAAIN, which starts with G.

Now, starting with G arrange the remaining 4 letters AAIN, (in which A occurs 2 times and the rest are different) in all possible ways.

Number of such arrangements $= \frac{4!}{2!} = 12.$

Thus, we obtain 12 new words.

The 37th word would be IAAGN, which starts with I.

Now, starting with I and arranging the remaining 4 letters, AAGN, we get $\frac{4!}{2!} = 12$ words.

The 49th word would be NAAGI and hence the 50th word is NAAIG.

EXAMPLE 11 *How many numbers greater than a million can be formed with the digits 2, 3, 0, 3, 4, 2, 3?*

SOLUTION Any number greater than one million will contain all the seven digits.

Now, we have to arrange seven digits, out of which 2 occurs twice, 3 occurs thrice and the rest are distinct.

Number of such arrangements $= \frac{7!}{(2!) \times (3!)} = 420.$

These arrangements will also include those which contain 0 at the million's place.

Keeping 0 fixed at the million's place, the remaining 6 digits out of which 2 occurs twice, 3 occurs thrice and the rest are distinct can be arranged in $= \frac{6!}{(2!) \times (3!)} = 60$ ways.

Hence, the number of required numbers $= (420 - 60) = 360$.

EXAMPLE 12 *How many numbers greater than* 400000 *can be formed by using the digits* 0, 2, 2, 4, 4, 5*?*

SOLUTION We are being given six digits and each number greater than 400000 is a six-digit number, having 4 or 5 in the extreme left position.

When 4 occupies the extreme left position, the remaining 5 places can be filled with the digits 0, 2, 2, 4, 5 (in which 2 occurs 2 times and rest are all different).

Number of such numbers $= \frac{5!}{2!} = 60$.

When 5 occupies the extreme left position, the remaining 5 places can be filled with the digits 0, 2, 2, 4, 4 (in which 2 occurs twice, 4 occurs twice and the rest are all distinct).

Number of such numbers $= \frac{5!}{(2!) \times (2!)} = 30$.

Hence, the required number of numbers $= (60 + 30) = 90$.

EXAMPLE 13 *How many numbers can be formed using the digits* 1, 2, 3, 4, 3, 2, 1 *so that the odd digits always occupy the odd places?*

SOLUTION We have been given seven digits, namely 1, 2, 3, 4, 3, 2, 1.

So, we have to form 7-digit numbers, so that odd digits occupy odd places.

Every 7-digit number has 4 odd places.

Given four odd digits are 1, 3, 3, 1 out of which 1 occurs 2 times and 3 occurs 2 times.

So, the number of ways to fill up 4 odd places $= \frac{4!}{(2!) \times (2!)} = 6$.

Given three even digits are 2, 4, 2 in which 2 occurs 2 times and 4 occurs 1 time.

So, the number of ways to fill up 3 even places $= \frac{3!}{2!} = 3$.

Hence, the required number of numbers $= (6 \times 3) = 18$.

EXERCISE 8E

1. Find the total number of permutations of the letters of each of the words given below:
   (i) APPLE (ii) ARRANGE (iii) COMMERCE
   (iv) INSTITUTE (v) ENGINEERING (vi) INTERMEDIATE
2. In how many ways can the letters of the expression $x^2y^3z^4$ be arranged when written without using exponents?
3. There are 3 blue balls, 4 red balls and 5 green balls. In how many ways can they be arranged in a row?
4. A child has three plastic toys bearing the digits 3, 3, 5 respectively. How many 3-digit numbers can he make using them?
5. How many different signals can be transmitted by arranging 2 red, 3 yellow and 2 green flags on a pole, if all the seven flags are used to transmit a signal?
6. How many words can be formed by arranging the letters of the word 'ARRANGEMENT', so that the vowels remain together?
7. How many words can be formed by arranging the letters of the word 'INDIA', so that the vowels are never together?
8. Find the number of arrangements of the letters of the word 'ALGEBRA' without altering the relative positions of the vowels and the consonants.
9. How many words can be formed from the letters of the word 'SERIES', which start with S and end with S?
10. In how many ways can the letters of the word 'PARALLEL' be arranged so that all L's do not come together?
11. How many different words can be formed with the letters of the word 'CAPTAIN'? In how many of these C and T are never together?
12. In how many ways can the letters of the word 'ASSASSINATION' be arranged so that all S's are together?
13. (i) How many arrangements can be made by using all the letters of the word 'MATHEMATICS'?
    (ii) How many of them begin with C?
    (iii) How many of them begin with T?
14. In how many ways can the letters of the word 'INTERMEDIATE' be arranged so that:
    (i) the vowels always occupy even places?
    (ii) the relative orders of vowels and consonants do not change?
15. (i) Find the number of different words formed by using all the letters of the word, 'INSTITUTION'.
    In how many of them
    (ii) are the three T's together?
    (iii) are the first two letters the two N's?
16. How many five-digit numbers can be formed with the digits 5, 4, 3, 5, 3?

17. How many numbers can be formed with the digits 2, 3, 4, 5, 4, 3, 2 so that the odd digits occupy the odd places?

18. How many 7-digit numbers can be formed by using the digits 1, 2, 0, 2, 4, 2, 4?

19. How many 6-digit numbers can be formed by using the digits 4, 5, 0, 3, 4, 5?

20. The letters of the word 'INDIA' are arranged as in a dictionary. What are the 1st, 13th, 25th, 49th and 60th words?

ANSWERS (EXERCISE 8E)

1. (i) 60 (ii) 1260 (iii) 5040 (iv) 30240 (v) 277200 (vi) 19958400

2. 1260 **3.** 27720 **4.** 3 **5.** 210 **6.** 60480

7. 42 **8.** 72 **9.** 12 **10.** 3000 **11.** 2520, 1800

12. 151200 **13.** (i) 4989600 (ii) 453600 (iii) 907200

14. (i) 21600 (ii) 21600 **15.** (i) 554400 (ii) 30240 (iii) 10080

16. 30 **17.** 18 **18.** 360 **19.** 150

20. (i) ADIIN (ii) DAIIN (iii) IADIN (iv) NADII (v) NIIDA

HINTS TO SOME SELECTED QUESTIONS

2. There are 2 x's, 3 y's and 4 z's.

$\therefore$ total number of arrangements $= \dfrac{9!}{(2!)\times(3!)\times(4!)}$.

3. We have to arrange 12 balls, out of which 3 are of one kind, 4 are of second kind and 5 are of third kind.

Required number of arrangements $= \dfrac{12!}{(3!)\times(4!)\times(5!)}$.

4. We have to arrange 3 objects in a row, out of which 2 are alike of one kind and the third one of its own kind.

Total number of arrangements $= \dfrac{3!}{2!} = 3$.

5. Required number of signals $= \dfrac{7!}{(2!)\times(3!)\times(2!)}$.

6. Let us assume that the vowels $\boxed{\text{AAEE}}$ form a single letter.

Now, $\boxed{\text{AAEE}}$ + RRNGMNT gives us 8 letters, out of which there are 2 R's, 2 N's, 1 G, 1 M and 1 T.

Number of these arrangements $= \dfrac{8!}{(2!)\times(2!)} = 10080$.

Now, AAEE has 4 letters, out of which there are 2 A's and 2 E's.

Number of their arrangements $= \dfrac{4!}{(2!)\times(2!)} = 6$.

Required number of words formed $= (10080 \times 6) = 60480$.

7. The word 'INDIA' contains 2 I's, 1 A, 1 N and 1 D.

Number of permutations of the letters of the given word $= \dfrac{5!}{2!} = 60$.

Let us assume that the vowels $\boxed{\text{IIA}}$ form 1 letter.

Now, $\boxed{IIA}$ + ND gives 3 letters, all distinct.
Number of their arrangements $= 3! = 6$.
Now, IIA has 3 letters, out of which there are 2 I's and 1 A.
Number of their arrangements $= \frac{3!}{2!} = 3$.
Number of words having all vowels together $= (6 \times 3) = 18$.
Number of words in which all vowels are not together $= (60 - 18) = 42$.

8. In the given word, there are 3 vowels, namely A, E, A out of which there are 2 A's and 1 E.
Number of arrangements of the vowels $= \frac{3!}{2!} = 3$.
The given word contains 4 consonants, namely L, G, B, R.
Number of arrangements of the consonants $= 4! = 24$.
Required number of arrangements $= (3 \times 24) = 72$.

9. After fixing S at the beginning and S at the end of each word, we have to arrange 4 letters out of which there are 2 E's, 1 R and 1 I.
Number of their arrangements $= \frac{4!}{2!} = 12$.

10. The given word 'PARALLEL' has 8 letters, out of which there are 2 A's, 3 L's, 1 P, 1 R and 1 E.
Number of their arrangements $= \frac{8!}{(2!) \times (3!)} = 3360$.
Let us assume $\boxed{LLL}$ as 1 letter.
Then, $\boxed{LLL}$ + PARAE has 6 letters, out of which there are 2 A's and the rest are all distinct.
Number of their arrangements $= \frac{6!}{2!} = 360$.
Number of arrangements in which 3 L's are not together $= (3360 - 360) = 3000$.

11. The given word 'CAPTAIN' has 7 letters, out of which there are 2 A's and the rest are all different.
Number of their arrangements $= \frac{7!}{2!} = 2520$.
Let us assume $\boxed{CT}$ as 1 letter.
Then, $\boxed{CT}$ + APAIN has 6 letters, out of which there are 2 A's and the rest are all distinct.
Number of their arrangements $= \frac{6!}{2!} = 360$.
Also, CT can be arranged among themselves in $2! = 2$ ways.
Number of arrangements in which C and T are together $= (360 \times 2) = 720$.
Number of arrangements in which C and T are never together $= (2520 - 720) = 1800$.

12. The given word 'ASSASSINATION' has 3 A's, 2 I's, 2 N's, 4 S's, 1 T and 1 O.
We consider $\boxed{SSSS}$ as one letter.
Then, $\boxed{SSSS}$ + AAAIINNTO consists of 10 letters, out of which there are 3 A's, 2 I's, 2 N's, 1 T, 1 O and 1 $\boxed{SSSS}$.
Number of their arrangements $= \frac{10!}{(3!) \times (2!) \times (2!)} = 151200$.

13. (i) The given word 'MATHEMATICS' has 11 letters, out of which there are 2 M's, 2 A's, 2 T's and the rest are all distinct.

$$\text{Number of their arrangements} = \frac{11!}{(2!)\times(2!)\times(2!)} = 4989600.$$

(ii) Let each arrangement begin with C. Then, out of the remaining 10 letters, there are 2 M's, 2 A's, 2 T's and the rest are all distinct.

$$\text{Number of all such arrangements} = \frac{10!}{(2!)\times(2!)\times(2!)} = 453600.$$

(iii) Let each arrangement begin with T. Then, out of the remaining 10 letters, there are 2 M's, 2 A's and the rest are all distinct.

$$\text{Number of all such arrangements} = \frac{10!}{(2!)\times(2!)} = 907200.$$

14. (i) The given word 'INTERMEDIATE' has 12 letters, namely 6 vowels and 6 consonants. Since vowels occupy even places, we have

$$\text{N} \quad \text{T} \quad \text{R} \quad \text{M} \quad \text{D} \quad \text{T} \quad =$$

type of arrangements, where vowels occupy even places, as shown by boxes.

It has 6 consonants, namely 2 T's, 1 N, 1 R, 1 M and 1 D.

$$\text{Number of arrangements of 6 consonants} = \frac{6!}{2!} = 360.$$

The given word has 6 vowels, namely 2 I's, 3 E's and 1 A.

$$\text{Number of arrangements of 6 vowels} = \frac{6!}{(2!)\times(3!)} = 60.$$

$\therefore$ total number of words formed $= (360 \times 60) = 21600.$

(ii) It is given that we may replace a vowel by another vowel and we can replace a consonant by a consonant, given herewith.

$$\text{Number of arrangements of consonants} = \frac{6!}{2!} = 360.$$

$$\text{Number of arrangements of vowels} = \frac{6!}{(3!)\times(2!)} = 60.$$

Total number of words formed $= (360 \times 60) = 21600.$

15. (i) The given word 'INSTITUTION' has 11 letters, out of which there are 3 I's, 3 T's, 2 N's, 1 S, 1 U and 1 O.

$$\text{Total number of words} = \frac{11!}{(3!)\times(3!)\times(2!)} = 554400.$$

(ii) Treating $\boxed{\text{TTT}}$ as one letter, the given word is expressed as $\boxed{\text{TTT}}$ + INSIUION, which consists of 9 letters out of which there are 3 I's, 2 N's, 1 S, 1 U and 1 O.

$$\text{Number of words in this case} = \frac{9!}{(3!)\times(2!)} = 30240.$$

(iii) Keeping $\boxed{\text{NN}}$ as first letter, we arrange the remaining 9 letters, out of which there are 3 I's, 3 T's, 1 S, 1 U and 1 O.

$$\text{Number of words formed} = \frac{9!}{(3!)\times(3!)} = 10080.$$

16. We have to form 5-digit numbers in which 5 occurs 2 times; 3 occurs 2 times and 4 occurs 1 time.

$$\text{Required number of numbers} = \frac{5!}{(2!)\times(2!)\times(1!)} = 30.$$

17. There are three odd digits, namely 3, 5, 3 out of which 3 occurs twice and 5 occurs once.

Number of ways to fill the odd places $= \frac{3!}{2!} = 3.$

There are four even digits namely 2, 4, 4, 2 out of which 2 occurs twice and 4 occurs twice.

Number of ways to fill the even places $= \frac{4!}{(2!)\times(2!)} = 6.$

Required number of numbers $= (3 \times 6) = 18.$

18. Required number of numbers $= \left\{\frac{7!}{(3!)\times(2!)} - \frac{6!}{(3!)\times(2!)}\right\} = (420 - 60) = 360.$

19. Required number of numbers $= \left\{\frac{6!}{(2!)\times(2!)} - \frac{5!}{(2!)\times(2!)}\right\} = (180 - 30) = 150.$

20. The word 'INDIA' contains 5 letters out of which there are 2 I's, 1 N, 1 D and 1 A.

Number of arrangements of its letters $= \frac{5!}{2!} = 60.$

In the dictionary, the first word is ADIIN.

Starting with A, we arrange the letters D, I, I, N in $\frac{4!}{2!} = 12$ ways.

The 13th word is DAIIN.

Starting with D, we arrange the letters A, I, I, N in $\frac{4!}{2!} = 12$ ways.

The 25th word is IADIN.

Starting with I, we arrange the letters A, D, I, N in $4! = 24$ ways.

The 49th word is NADII.

Starting with N, we arrange the letters A, D, I, I in $\frac{4!}{2!} = 12$ ways.

The 60th word is NIIDA.

PERMUTATIONS WITH REPETITIONS

THEOREM 1 *The number of permutations of n different objects, taken r at a time when each may be repeated any number of times in each arrangement, is* n^r.

PROOF We know that the number of all permutations of *n* objects, taken *r* at a time is the same as the number of ways of filling up *r* places with *n* different objects.

Clearly, the first place can be filled in *n* ways. Since each object may be repeated, the second place can be filled in *n* ways.

Similarly, each of the 3rd, 4th, ..., *r*th places can be filled in *n* ways.

$\therefore$ the requisite number of permutations $= n \times n \times n \times \ldots r$ times $= n^r$.

Corollary When $r = n$,

i.e., the number of permutations of *n* different objects, taken all at a time when each may be repeated any number of times in each arrangement, is n^n.

SUMMARY

(i) Number of permutations of n different objects, taken r at a time when each may be repeated any number of times in each arrangement, is n^r.

(ii) Number of permutations of n different objects, taken all at a time when each may be repeated any number of times in each arrangement, is n^n.

SOLVED EXAMPLES

EXAMPLE 1 *In how many ways can* 4 *letters be posted in* 3 *letter boxes?*

SOLUTION Each letter can be posted in any of the 3 letter boxes.
So, each letter can be posted in 3 ways.
$\therefore$ 4 letters can be posted in $(3 \times 3 \times 3 \times 3) = 3^4 = 81$ ways.

EXAMPLE 2 *In how many ways can* 6 *different rings be worn in* 4 *fingers of the hand?*

SOLUTION Each ring may be worn in any of the 4 fingers.
So, each ring may be worn in 4 ways.
$\therefore$ 6 rings may be worn in $(4 \times 4 \times 4 \times 4 \times 4 \times 4) = 4^6 = 4096$ ways.

EXAMPLE 3 *In how many ways can* 5 *apples be distributed among* 6 *boys, there being no restriction to the number of apples each boy may get?*

SOLUTION Each apple may be given to any of the 6 boys.
So, each apple may be given in 6 ways.
$\therefore$ 5 apples may be given in $(6 \times 6 \times 6 \times 6 \times 6) = 6^5 = 7776$ ways.

EXAMPLE 4 *In how many ways can* 3 *prizes be distributed among* 4 *boys, when*

(i) no boy gets more than 1 *prize;*

(ii) a boy may get any number of prizes;

(iii) no boy gets all the prizes?

SOLUTION **Case (i)** *When no boy gets more than* 1 *prize.*
The first prize may be given to any of the 4 boys in 4 ways.
The 2nd prize may be given in 3 ways, since the boy who got the 1st prize cannot receive it.
The 3rd prize may be given to any of the remaining 2 boys in 2 ways.
Required number of ways $= (4 \times 3 \times 2) = 24$.

Case (ii) *When a boy may get any number of prizes.*
The 1st prize may be given to any of the 4 boys in 4 ways.
Since a boy may get any number of prizes, so 2nd prize may be given to any of the 4 boys in 4 ways.
Similarly, the 3rd prize may be given in 4 ways.
Required number of ways $= (4 \times 4 \times 4) = 64$.

Case (iii) *When no boy gets all the prizes.*

The number of ways in which a boy gets all the prizes is 4, as anyone of the 4 boys may get all the prizes.

Hence, the number of ways in which a boy does not get all the prizes = $(64 - 4) = 60$.

EXAMPLE 5 *How many four-digit numbers can be formed with the digits* 1, 2, 3, 4, 5, 6 *when a digit may be repeated any number of times in any arrangement?*

SOLUTION Clearly, the thousand's place can be filled in 6 ways as anyone of the 6 digits may be placed at it.

Since there is no restriction on repetition of digits, each one of the hundred's, ten's and unit's digits can be filled in 6 ways.

Hence, the required number of numbers = $(6 \times 6 \times 6) = 216$.

EXAMPLE 6 *How many 4-digit numbers are there when a digit may be repeated any number of times?*

SOLUTION Clearly, 0 cannot be placed at the thousand's place.

So, this place can be filled with any digit from 1 to 9.

Thus, there are 9 ways of filling the thousand's place.

Since repetition of digits is allowed, each one of the remaining 3 places can be filled in 10 ways, i.e., with any digit from 0 to 9.

So, the required number of numbers = $(9 \times 10^3) = 9000$.

EXERCISE 8F

1. A child has 6 pockets. In how many ways can he put 5 marbles in his pocket?
2. In how many ways can 5 bananas be distributed among 3 boys, there being no restriction to the number of bananas each boy may get?
3. In how many ways can 3 letters be posted in 2 letter boxes?
4. How many 3-digit numbers are there when a digit may be repeated any number of times?
5. How many 4-digit numbers can be formed with the digits 0, 2, 3, 4, 5 when a digit may be repeated any number of times in any arrangement?
6. In how many ways can 4 prizes be given to 3 boys when a boy is eligible for all prizes?
7. There are 4 candidates for the post of a chairman and one is to be elected by votes of 5 men. In how many ways can the votes be given?

ANSWERS (EXERCISE 8F)

1. 6^5 **2.** 3^5 **3.** 8 **4.** 900 **5.** 500 **6.** 81 **7.** 1024

2. Each banana may be given to any of the 3 boys.
   So, each banana can be given in 3 ways.
3. Each letter may be posted in 2 ways.
4. We may fill hundred's digit by any digit from 1 to 9, i.e., in 9 ways. Each of the ten's and unit's digits may be filled in 10 ways.
   Required number of numbers $= (9 \times 10 \times 10) = 900$.
5. Required number of numbers $= (4 \times 5 \times 5 \times 5) = 500$.
6. Each prize can be distributed in 3 ways.
7. Each man can vote in 4 ways.

CIRCULAR PERMUTATIONS

So far we have been considering the arrangements of objects in a line. Such permutations are known as linear permutations.

Instead of arranging the objects in a line, if we arrange them in the form of a circle, we call them *circular permutations*.

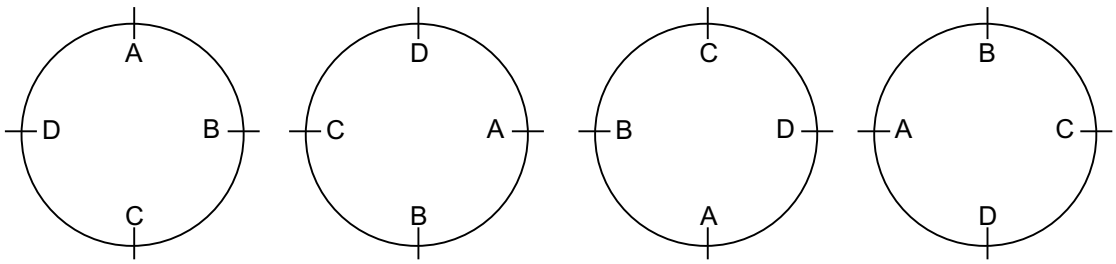
In circular permutations, what really matters is the position of an object relative to the others.

If we consider the linear permutations

ABCD, BCDA, CDAB and *DABC*

then, clearly, they are distinct.

Now, we arrange *A, B, C, D* along the circumference of a circle as shown below:



If we consider the position of an object relative to others then we find that the above four arrangements are the same.

For example, if four persons A, B, C, D sit round a table as shown in the first arrangement and then each one shifts to the next chair to the left, we obtain the second arrangement.

In this arrangement each person has the same neighbour to the left, and the same neighbour to the right, as he had in the first arrangement.

Thus, the first and the second arrangements give rise to the same circular permutation.

Same is the case with the third and fourth arrangements given above.

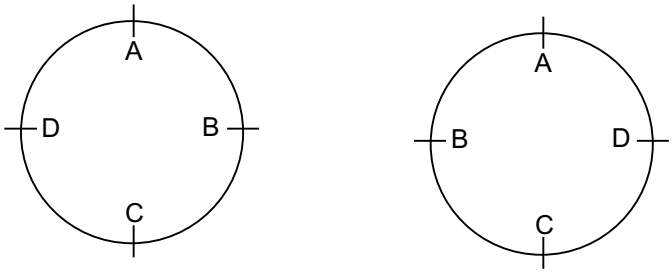
Thus, in circular permutations, we fix the position of one of the objects and then arrange the other objects in all possible ways.

There are two types of circular permutations:

(i) The circular permutations in which the clockwise and the anticlockwise arrangements give rise to different permutations, e.g., seating arrangements of persons round a table.

(ii) The circular permutations in which the clockwise and the anticlockwise arrangements give rise to same permutations, e.g., arranging some beads to form a necklace.

Look at the circular permutations, given below:



Suppose A, B, C, D are the four beads forming a necklace.

They have been arranged in clockwise and anticlockwise directions in the first and second arrangements respectively.

Now, if the necklace in the first arrangement be given a turn, from clockwise to anticlockwise, we obtain the second arrangement. Thus, there is no difference between the above two arrangements.

RESULTS ON CIRCULAR PERMUTATIONS

THEOREM 1 *The number of circular permutations of n different objects is* $(n-1)!$.

PROOF Fixing the position of an object can be done in *n* ways, as the position of anyone of them may be fixed.

Thus, each circular permutation corresponds to *n* linear permutations, depending upon where from we start.

Since there are $n!$ linear permutations, it follows that there are $\frac{n!}{n}$, i.e., $(n-1)!$ circular permutations.

THEOREM 2 *The number of ways in which n persons can be seated round a table is* $(n-1)!$.

PROOF Let us fix the position of one person and then arrange the remaining $(n-1)$ persons in all possible ways. Clearly, this can be done in $(n-1)!$ ways.

Hence, the required number of ways = $(n-1)!$.

THEOREM 3 *Show that the number of ways in which n different beads can be arranged to form a necklace is* $\frac{1}{2}(n-1)!$.

PROOF Fixing the position of one bead, the remaining $(n-1)$ beads can be arranged in $(n-1)!$ ways.

In case of arranging the beads, there is no distinction between the clockwise and the anticlockwise arrangements.

So, the required number of ways $= \frac{1}{2}(n-1)!$.

ILLUSTRATIVE EXAMPLES

EXAMPLE 1 *In how many ways can 8 students be arranged in*

(i) a line, (ii) a circle?

SOLUTION (i) The number of ways in which 8 students can be arranged in a line $= 8! = (8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1) = 40320$.

(ii) The number of ways in which 8 students can be arranged in a circle $= (8-1)! = 7! = (7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1) = 5040$.

EXAMPLE 2 *If 20 persons were invited for a party, in how many ways can they and the host be seated at a circular table?*
In how many of these ways will two particular persons be seated on either side of the host?

SOLUTION Clearly, there are 21 persons, to be seated round a circular table.

(i) Let us fix the seat of one person, say the host. The remaining 20 persons can now be arranged in 20 ! ways.

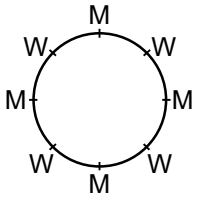
Hence, the number of ways in which these 21 persons can be seated round a circular table = 20 !.

(ii) The two particular persons can be seated on either side of the host in 2 ways and for each way of their taking seats, the remaining 18 persons can be seated at the circular table in 18 ! ways.

$\therefore$ the number of ways of seating 21 persons at a circular table with two particular persons on either side of the host $= (2 \times 18!)$.

EXAMPLE 3 *In how many ways can a party of 4 men and 4 women be seated at a circular table so that no two women are adjacent?*

SOLUTION The 4 men can be seated at the circular table such that there is a vacant seat between every pair of men.



The number of ways in which these 4 men can be seated at the circular table = 3 ! = 6.

Now, the 4 vacant seats may be occupied by 4 women in

$^4P_4 = 4! = 24$ ways.

$\therefore$ the required number of ways $= (6 \times 24) = 144$.

EXAMPLE 4 *A round table conference is to be held between delegates of 20 countries. In how many ways can they be seated if two particular delegates may wish to sit together?*

SOLUTION Regarding these two delegates as one, the 19 delegates can be arranged at a circular table in 18 ! ways.

These two delegates can be arranged among themselves in 2 !, i.e., 2 ways.

$\therefore$ the required number of ways $= 2 \times (18!)$.

EXAMPLE 5 *In how many ways can 7 persons sit around a table so that all shall not have the same neighbours in any two arrangements?*

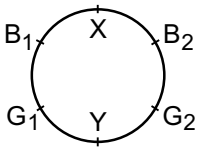
SOLUTION 7 persons sit at a round table in 6 ! ways.

But, in clockwise and anticlockwise arrangements, each person will have the same neighbours.

So, the required number of ways $= (1/2) \times (6!) = 360$.

EXAMPLE 6 *3 boys and 3 girls are to be seated around a table in a circle. Among them, the boy X does not want any girl neighbour and the girl Y does not want any boy neighbour. How many such arrangements are possible?*

SOLUTION Fix the positions of X and Y as shown in the figure.



Now, the boys B_1, B_2 and the girls G_1, G_2 may have their neighbours X and Y respectively.

The boys B_1 and B_2 may be arranged among themselves in 2 ! ways. And, the girls G_1 and G_2 may be arranged among themselves in 2 ! ways.

Hence, the required number of arrangements $= (2 \times 2) = 4$.

EXAMPLE 7 *Find the number of ways in which 8 different beads can be arranged to form a necklace.*

SOLUTION Fixing the position of one bead, the remaining beads can be arranged in 7! ways. But, there is no distinction between the clockwise and anticlockwise arrangements.

So, the required number of arrangements $= (1/2) \times (7!) = 2520$.

EXERCISE 8G

1. In how many ways can 6 persons be arranged in
   (i) a line, (ii) a circle?
2. There are 5 men and 5 ladies to dine at a round table. In how many ways can they seat themselves so that no two ladies are together?
3. In how many ways can 11 members of a committee sit at a round table so that the secretary and the joint secretary are always the neighbours of the president?
4. In how many ways can 8 persons be seated at a round table so that all shall not have the same neighbours in any two arrangements?

5. In how many different ways can 20 different pearls be arranged to form a necklace?
6. In how many different ways can a garland of 16 different flowers be made?

ANSWERS (EXERCISE 8G)

1. (i) $6! = 720$ (ii) 120
2. $(4!) \times (5!) = 2880$
3. 80640
4. 2520
5. $\frac{1}{2} \times (19!)$
6. $\frac{1}{2} \times (15!)$

EXERCISE 8H

Very-Short-Answer Questions

1. If $(n+1)! = 12 \times [(n-1)!]$, find the value of n.
2. If $\frac{1}{4!} + \frac{1}{5!} = \frac{x}{6!}$, find the value of x.
3. How many 3-digit numbers are there with no digit repeated?
4. How many 3-digit numbers above 600 can be formed by using the digits 2, 3, 4, 5, 6, if repetition of digits is allowed?
5. How many numbers divisible by 5 and lying between 4000 and 5000 can be formed from the digits 4, 5, 6, 7, 8, if repetition of digits is allowed?
6. In how many ways can the letters of the word 'CHEESE' be arranged?
7. In how many ways can the letters of the word 'PERMUTATIONS' be arranged if each word starts with P and ends with S?
8. How many different words can be formed by using all the letters of the word 'ALLAHABAD'?
9. How many permutations of the letters of the word 'APPLE' are there?
10. How many words can be formed by the letters of the word 'SUNDAY'?
11. In how many ways can 4 letters be posted in 5 letter boxes?
12. In how many ways can 4 women draw water from 4 taps, if no tap remains unused?
13. How many 5-digit numbers can be formed by using the digits 0, 1 and 2?
14. In how many ways can 5 boys and 3 girls be seated in a row so that each girl is between 2 boys?
15. A child has plastic toys bearing the digits 4, 4 and 5. How many 3-digit numbers can he make using them?
16. In how many ways can the letters of the word 'PENCIL' be arranged so that N is always next to E?

ANSWERS (EXERCISE 8H)

1. $n = 3$	2. $x = 36$	3. 648	4. 25	5. 25
6. 120	7. 1814400	8. 7560	9. 60	10. 720

11. 625 **12.** 24 **13.** 162 **14.** 24 **15.** 3
16. 120

HINTS TO SOME SELECTED QUESTIONS

1. $(n+1)\times n\times[(n-1)!] = 12\times[(n-1)!] \Rightarrow (n+1)\times n = 12 = 4\times 3 \Rightarrow n = 3.$

2. $\frac{1}{4!}+\frac{1}{5\times(4!)} = \frac{x}{(6\times 5)\times(4!)} \Rightarrow \frac{x}{30} = \left(1+\frac{1}{5}\right) = \frac{6}{5} \Rightarrow x = \left(\frac{6}{5}\times 30\right) = 36.$

3. The hundred's, ten's and unit's digits can be filled in 9, 9 and 8 ways respectively.
Hence, the required number of numbers $= (9\times 9\times 8) = 648.$

4. Given digits are 2, 3, 4, 5, 6 and repetition of digits is allowed.
The hundred's place can be filled by 6 in 1 way only.
Each of the ten's and unit's digits can be filled by any of the given five digits in 5 ways.
Required number of numbers $= (1\times 5\times 5) = 25.$

5. Given digits are 4, 5, 6, 7, 8 and repetition of digits is allowed.
Clearly, the required number must have 4 at the thousand's place.
So, this place can be filled in 1 way.
Since each number is divisible by 5, so we must have 5 at the unit's place. So, unit's place can be filled in 1 way.
Each of the hundred's and ten's places can be filled in 5 ways.
$\therefore$ the required number of numbers $= (1\times 1\times 5\times 5) = 25.$

6. The given word contains 6 letters, out of which E is repeated 3 times and all others are different.
So, the required number of ways $= \frac{6!}{3!} = 120.$

7. The given word contains 12 letters. Keeping P and S at first and last positions, 10 letters remain there out of which, there are 2 T's and the rest are different.
Required number of ways $= \frac{10!}{2!} = 1814400.$

8. The given word contains 9 letters out of which there are 4 A's, 2 L's and others are all distinct.
Required number of words $= \frac{9!}{(4!)\times(2!)} = 7560.$

9. The given word contains 5 letters, out of which there are 2 P's and the rest are all different.
Required number of permutations $= \frac{5!}{2!} = 60.$

10. The given word contains 6 letters, all distinct.
Required number of words $= 6! = 720.$

11. Every letter can be posted in 5 ways.
Required number of ways $= 5^4.$

12. 1st tap can be used by any of the 4 women in 4 ways.
2nd tap can be used by any of the remaining 3 women.

3rd tap can be used by any of the remaining 2 women.
4th tap can be used by the remaining 1 woman.
Required number of ways $= 4 \times 3 \times 2 \times 1 = 24$.

13. The ten-thousand's place can be filled by 1 or 2, i.e., in 2 ways.
Each of the other 4 digits can be filled in 3 ways.
Required number of numbers $= (2 \times 3 \times 3 \times 3 \times 3) = 162$.

14. Let us fix 5 boys, keeping a space between every 2 boys, as shown below:

$$B = B = B = B = B.$$

Required number of ways $= {}^4P_3 = (4 \times 3 \times 2) = 24$.

15. There are 3 digits out of which 4 is repeated 2 times and 5 occurs only once.

Required number of numbers $= \frac{3!}{2!} = \frac{3 \times 2 \times 1}{2} = 3$.

16. Keeping $\boxed{EN}$ as 1 letter, this letter and 4 other letters of the given word can be arranged among themselves in $5! = 120$ ways.

KEY FACTS AND FORMULAE

1. (i) $n! = n \times (n-1) \times (n-2) \times \ldots \times 3 \times 2 \times 1$

(ii) $n! = n \times [(n-1)!]$

(iii) $0! = 1$

2. Fundamental Principle of Multiplication

If there are two operations such that one of them can be performed in m ways and when it has been performed in anyone of these m ways, the second operation can be performed in n ways, then the two operations in succession can be performed in $(m \times n)$ ways.

3. Fundamental Principle of Addition

If there are two operations such that they can be performed independently in m and n ways respectively, then either of the two operations can be performed in $(m + n)$ ways.

4. Permutations The different arrangements which can be made out of a given number of things by taking some or all at a time, are called permutations.

5. (i) Number of permutations of n different objects taking r at a time, where $0 < r \leq n$ and the objects do not repeat, is given by

$${}^nP_r = n(n-1)(n-2)\ldots(n-r+1) = \frac{n!}{(n-r)!}.$$

(ii) ${}^nP_n = n!$.

(iii) Let there be n things of which m are alike of one kind and the remaining $(n-m)$ are alike of second kind, then

$$\text{total number of permutations} = \frac{n!}{(m!) \times [(n-m)!]}.$$

(iv) The number of permutations of n objects, where p_1 objects are of one kind, p_2 are of second kind, ..., p_k are of kth kind and the rest, if any, are distinct, is

$$\frac{n!}{(p_1!)\times(p_2!)\times\ldots\times(p_k!)}.$$

(v) Number of permutations of n different objects, taken r at a time when each may be repeated any number of times in each arrangement, is n^r.

6. The number of circular permutations of n different objects is $(n-1)!$.

7. The number of ways in which n persons can be seated round a table is $(n-1)!$.

8. The number of ways in which n different beads can be arranged to form a necklace is $\frac{1}{2}(n-1)!$.